LINEAR ALGEBRA

PART I

THE LINEAR ALGEBRA

§ 1.1. *n* – dimensional vectors

*Definition***.** A set of n numbers is said to be a **vector.**

$$\overline{a} = (a_1, a_2, \ldots, a_n), \text{ or } \overline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Numbers $a_1, a_2, \ldots, a_n$ are called **coordinates** of a vector $\overline{a}$.

Definition. Two vectors are equal if their coordinates are equal:

$$\overline{a} = \overline{b} \Leftrightarrow \begin{cases} a_1 = b_1 \\ \vdots \\ a_n = b_n \end{cases} \qquad (1.1.1)$$

Operations on vectors

1. Multiplication of a vector $\overline{a}$ by a scalar λ:

$$\lambda \, \overline{a} = \lambda \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}. \qquad (1.1.2)$$

2. The sum (difference) of vectors.

Let the vectors $\overline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\overline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ be given then

$$\overline{a} \pm \overline{b} = \begin{pmatrix} a_1 \pm b_1 \\ \vdots \\ a_n \pm b_n \end{pmatrix} \qquad (1.1.3)$$

§ 1.2. Linear Dependence of Vectors

Let us have a set of *n*-dimensional vectors

$$\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n. \qquad (1.2.1)$$

*Definition***.** The expression

$$\lambda_1 \bar{a}_1 + \lambda_2 \bar{a}_2 + \ldots + \lambda_n \bar{a}_n \qquad (1.2.2)$$

is called a **linear combination** of the vectors (1.2.1).

*Definition***.** If one of the vectors (1.2.1) is a linear combination of the remaining vectors then a set of vectors (1.2.1) is called a **linear dependent set** of vectors.

*Definition***.** A linear combination of vectors (1.2.2) is said to be **trivial** if all its coefficients equal zero: $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.
If at least one of $\lambda_i \neq 0$ then (1.2.2) is called **non-trivial combination.**

Theorem. The vectors (1.2.1) are linear dependent if and only if there exists a non-trivial combination of these vectors equals zero.

1. Let vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ be linear dependent then one of these vectors is a linear combination of the remaining vectors. For example this vector is $\bar{a}_2$. So we have

$$\bar{a}_2 = \lambda_1 \bar{a}_1 + \lambda_3 \bar{a}_3 + \ldots \lambda_n \bar{a}_n \Rightarrow$$
$$\Rightarrow \lambda_1 \bar{a}_1 - \bar{a}_2 + \lambda_3 \bar{a}_3 + \ldots + \lambda_n \bar{a}_n = 0$$

where $\lambda_2 = -1 \neq 0$. It means that the linear combination (2.2) is non-trivial.

2. Now there exists non-trivial combination

$$\lambda_1 \bar{a}_1 + \lambda_2 \bar{a}_2 + \ldots + \lambda_n \bar{a}_n = 0,$$

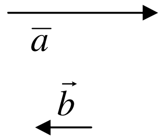
where $\lambda_2 \neq 0$, so

$$\bar{a}_2 = -\frac{\lambda_1}{\lambda_2} \bar{a}_1 - \ldots - \frac{\lambda_n}{\lambda_2} \bar{a}_n$$

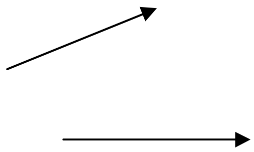
As we see $\bar{a}_2$ is a linear combination of the rest vectors. The theorem is proved.

Example.

1. Collinear vectors are linear dependent vectors. In fact, as we know



$\overline{a} \| \overline{b} \Leftrightarrow \overline{b} = \lambda\overline{a}$, so we have that $\lambda\overline{a} - \overline{b} = 0$.

2.The vectors  are linear independent.

§ 1.3. Matrices

Definition. A rectangular array of numbers is called a **matrix.**

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

This matrix has *m ro*ws and *n* columns. We call ***A*** a "*m* by *n*" matrix or a matrix of [*mxn*] dimension.

The element in the *i*-th row and *j*-th column of a matrix can be represented by a_{ij}.

We denote matrices by letters ***A, B, C*** and their elements by a_{ij}, b_{ij}, c_{ij}.

$A = (a_{ij}), B = (b_{ij}), \ C = (c_{ij})$.

If $m = n$ then a matrix is called a **square matrix**. It is called a matrix of order *n*, for short.

Two matrices ***A*** and ***B*** are **equal** if and only if they have the same elements in the same positions. For example, ***A = B*** if they are of one and the same dimension and $a_{ij} = b_{ij}$ for any *i* and *j*.

If we interchange columns and rows of a matrix *A* we get the **transposed** matrix A^T :

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \cdots & \cdots & \cdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}. \qquad (1.3.1)$$

For example,

$$\begin{pmatrix} 2 & 6 & 3 \\ 9 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & 9 \\ 6 & 1 \\ 3 & 0 \end{pmatrix}.$$

Let the square matrix A be given. The diagonal containing $a_{11}, a_{22}, \ldots, a_{n-1n-1}, a_{nn}$ is called the **principal** (main) **diagonal.**

*Definition.*If there are nonzero elements on the main diagonal of a square matrix A and zeroes elsewhere then this matrix is called a **diagonal matrix.**

Definition. A diagonal matrix is said to be a **unit-matrix** if all diagonal elements equal 1. It is denoted by I or E.

$$I = E = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \qquad (1.3.2)$$

Definition..Matrix, all elements of which situated under (over) its principal diagonal is called a **triangular** matrix.

There are two following examples of the triangular matrices:

$$\begin{pmatrix} 2 & -7 & 0 & 1 \\ 0 & 6 & 9 & -4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 4 & 5 \end{pmatrix}$$

§ 1.4. Determinants

With square matrix of order n we associate a number called the **determinant of A** and written sometimes det$A$ and sometimes $|A|$ with vertical bars (which do not mean absolute value). For $n = 1$ and $n = 2$ we have these definitions:

$$\det a_{11} = a_{11} \tag{1.4.1}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}. \tag{1.4.2}$$

We introduce for a matrix of order $n \geq 3$, first of all, these definitions:

Definition.. A **minor** M_{ij} of an element a_{ij} of a matrix A is a determinant obtained from a given matrix deleting the i-th row and j-th column.

Definition.. A quantity $(-1)^{i+j}M_{ij}$ is called a **cofactor** A_{ij} of an element a_{ij}

Example 1.4.1. Calculate M_{23} and A_{23} of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 5 & 3 \\ 7 & 3 & 5 \end{pmatrix}$.

$$M_{23} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 5 & 3 \\ 7 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 3 \end{vmatrix} = 3 - 14 = -11.$$

$$A_{23} = (-1)^{2+3}M_{23} = -(-11) = 11.$$

Theorem. The sum of the products of elements a_{ij} of any row (column) of a determinant and their cofactors is equal to one and the same number. This number is a value of the given determinant.

Example 1.4.2. Calculate the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 5 & 3 \\ 7 & 3 & 5 \end{pmatrix}$.

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 5 & 3 \\ 7 & 3 & 5 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1 \cdot \begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix} - 2 \cdot \begin{vmatrix} 5 & 3 \\ 7 & 5 \end{vmatrix} + 3 \cdot \begin{vmatrix} 5 & 5 \\ 7 & 3 \end{vmatrix} =$$

$$= 16 - 8 - 60 = -52.$$

The expression $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ is called **the expanding determinant by the first row.**

Note that we can calculate determinants of the third order using the following rule:

1. supplement the first and the second rows accordingly,
2. take the sum of the products of the elements of the main diagonal and of the its parallel,
3. subtract the products of the elements of the order diagonal and its parallel.

   Using this rule we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 5 & 5 & 3 \\ 7 & 3 & 5 \end{vmatrix} \begin{matrix} \\ \\ \\ 1 & 2 & 3 \\ 5 & 5 & 3 \end{matrix} = 1 \cdot 5 \cdot 5 + 5 \cdot 3 \cdot 3 + 7 \cdot 2 \cdot 3 - (7 \cdot 5 \cdot 3 + 1 \cdot 3 \cdot 3 + 5 \cdot 2 \cdot 5) =$$

= 25 + 45 + 42 – 105 – 9 – 50 = 52.

We now state some properties of determinants. You should know and be able to use these facts, but we omit the proofs.

1. The determinant of the transposed matrix is equal to the given determinant:

   $$|A^T| = |A|.$$

2. If two rows (columns) of a determinant are identical (or are proportional), the determinant is zero.

3. If two rows (columns) of a determinant are interchanged, the determinant just changes its sign.

4. If each element of some row (column) of a determinant is multiplied by a constant λ, the determinant is multiplied by λ:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ \lambda a_{n1} & \cdots & \lambda a_{nn} \end{vmatrix} = \lambda \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}.$$

5. $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21}+b_1 & a_{22}+b_2 & \cdots & a_{2n}+b_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ b_1 & b_2 & \cdots & b_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

6. If all elements of a determinant above the principal diagonal (or all below it) are zero, the determinant is the product of the elements of the main diagonal.

For example,

$$\begin{vmatrix} 2 & 0 & 0 \\ 5 & -2 & 0 \\ 1 & 4 & 7 \end{vmatrix} = (2)(-2)(7) = -28,$$

7. If each element of a row (or column) is multiplied by a constant c and the results are added to a different row (or column), the determinant is not changed.

For example,

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix} \begin{matrix} \\ II-2I \\ III+I \\ \\ \end{matrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 5 & -6 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 1 & 2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 5 & -6 & 0 \end{vmatrix} \begin{matrix} \\ \\ \\ \text{IY}-5\text{II} \end{matrix} =$$

$$= -\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -16 & -5 \end{vmatrix} \begin{matrix} \\ \\ \\ \text{IY}+4\text{III} \end{matrix} = -\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & -9 \end{vmatrix} = -(-36) = 36$$

§ 1.5. Operations with Matrices

•1. Multiplying by a number.

To multiply a matrix A by a number λ we multiply each element of this matrix by λ:

$$A = (a_{ij}) \Leftrightarrow \lambda A = (\lambda a_{ij}) \qquad (1.5.1)$$

Example.

$$A = \begin{pmatrix} 3 & 5 & -1 \\ \frac{3}{4} & -2 & 2 \end{pmatrix}, \text{ then } 4A = \begin{pmatrix} 12 & 20 & -4 \\ 3 & -8 & 8 \end{pmatrix}.$$

• 2. Addition (Subtraction) of Matrices.

We can add and subtract the matrices of one and the same dimension. Their sum (difference) is the matrix we get by adding (subtracting) corresponding elements in the given matrices:

$$A \pm B = (a_{ij} \pm b_{ij}). \qquad (1.5.2)$$

Example. Let the matrices

$$A = \begin{pmatrix} 2 & -7 \\ 5 & 4 \end{pmatrix}, \ B = \begin{pmatrix} -3 & 1 \\ 5 & 0 \end{pmatrix} \quad \text{be given.}$$

Find: 1) their sum, 2) difference $B - A$.

Solution.

1) $A + B = \begin{pmatrix} 2-3 & -7+1 \\ 5+5 & 4+0 \end{pmatrix} = \begin{pmatrix} -1 & -6 \\ 10 & 4 \end{pmatrix}$;

2) $B - A = \begin{pmatrix} -3-2 & 1+7 \\ 5-5 & 0-4 \end{pmatrix} = \begin{pmatrix} -5 & 8 \\ 0 & -4 \end{pmatrix}$.

- **3. Multiplication of Matrices.**

a) The first step.
Multiplication of a matrix-row by a matrix-column (n-tuple row by n-tuple column)

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \cdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i$$ - the sum of products of the corresponding elements.

For example,

$$\begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} = 2 \cdot (-2) - 1 \cdot 0 + 3 \cdot 4 = 8.$$

b) The second step.
Multiplication of a matrix $[mxn]$ A by n-tuple column B. To form this product (a matrix C), we take the elements of the first row of A in the order from left to right and multiply by the corresponding of a matrix B. This is the first row of C. Then we repeat the process using the second row of A. For example,

$$\begin{pmatrix} 2 & -1 & 3 \\ 4 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y + 3z \\ 4x - 5z \end{pmatrix}.$$

c) The third step.

The multiplication a matrix A[*m*x*n*] by a matrix B[*n*x*p*] is a matrix C[*m*x*p*] whose element c_{ij} is the product of the *i*-th row of *A* and the *j*-th column of *B:*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \ i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, p.$$

For example,

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix} = \begin{pmatrix} 2a + 3d - x & 2b + 3e - y & 2c + 3f - z \\ 4a - 2d + 5x & 4b - 2e + 5y & 4c - 2f + 5z \end{pmatrix}.$$

Properties of the Product of Matrices

1. In general $AB \neq BA$.

Examples.

a) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$$AB = \begin{pmatrix} 2+4 & 4+3 \\ 1+8 & 2+6 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 9 & 8 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2+2 & 1+4 \\ 8+3 & 4+6 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 11 & 10 \end{pmatrix}.$$

As we see in this case $AB \neq BA$.

b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ -5 & 2 \\ 7 & 1 \end{pmatrix}$

It is impossible to calculate AB, but it is possible to multiply B by A:

$$BA = \begin{pmatrix} 0 & 3 \\ -5 & 2 \\ 7 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0+3 & 0+6 \\ -10+2 & -5+4 \\ 14+1 & 7+2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -8 & -1 \\ 15 & 9 \end{pmatrix}.$$

c) $A = \begin{pmatrix} 2 & -1 & 9 \\ 3 & 5 & 3 \\ 1 & 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$AB = \begin{pmatrix} 2 & -1 & 9 \\ 3 & 5 & 3 \\ 1 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 18 \\ 6 & 10 & 6 \\ 2 & 8 & 2 \end{pmatrix},$$

$$BA = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 9 \\ 3 & 5 & 3 \\ 1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 18 \\ 6 & 10 & 6 \\ 2 & 8 & 2 \end{pmatrix}.$$

In this case $AB = BA$.

2. $(AB)C = A(BC)$

3. $(A+B)C = AC + BC$ or $C(A + B) = CA + CB$

4. Let A and B be square matrices of the same order, then

$$\det(AB) = \det A \cdot \det B.$$

§ 1.6. The Inverse of a Square Matrix

Definition. A square matrix B is said to be an inverse matrix of A if $AB = BA = I$ and it is denoted by the symbol A^{-1}. So we have

$$AA^{-1} = A^{-1}A = I \qquad (1.6.1)$$

Definition. Transposed matrix of cofactors of the corresponding elements of the given matrix A is called the **adjoint** of A:

$$\text{adj}\, A = \widetilde{A} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}^T \qquad (1.6.2)$$

Example 1.6.1.

Find the adjoint of the matrix A if

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution.

$$A_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2; \qquad A_{12} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1; \qquad A_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1;$$

$$A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1; \qquad A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0; \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1;$$

$$A_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5; \qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1; \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3.$$

So we have

$$\tilde{A} = \begin{pmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ 5 & -1 & -3 \end{pmatrix}^T = \begin{pmatrix} -2 & -1 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix}.$$

Check up that

$$\tilde{A}A = A\tilde{A} = \Delta \cdot I = \det A \cdot I \qquad (1.6.3)$$

Theorem. A matrix *A* has an inverse if and only if its determinant is not equal to zero.

1. Let $\det A = |\,A\,| \neq 0$ be given. Prove A^{-1} exists.

Let us use the equality (1.6.3):

$$\tilde{A}A = A\tilde{A} = \Delta \cdot I = \det A \cdot I \Rightarrow A\tilde{A} = \Delta \cdot I$$

but $\Delta \neq 0$, so

$$\left(\frac{\tilde{A}}{\Delta}\right)A = I, \text{ or } A\left(\frac{\tilde{A}}{\Delta}\right) = I.$$

It means that

$$A^{-1} = \frac{\tilde{A}}{\Delta} \qquad (1.6.4)$$

2. A^{-1} exists, prove that $\Delta \neq 0$. In fact,

$$A^{-1}A = I \Rightarrow \det\left(A^{-1}A\right) = \det I \Rightarrow \det A^{-1} \det A = 1 \Rightarrow$$
$$\Rightarrow \det A^{-1} \cdot \Delta = 1 \Rightarrow \Delta \neq 0.$$

The theorem is proved.

The formula (1.6.4) gives the method of finding the inverse matrix:

1. be sure that $\Delta = |\,A\,| \neq 0$,
2. construct the matrix of corresponding cofactors and transpose it,
3. divide this matrix by $|\,A\,|$.

Example 1.6.2. Find the inverse of A if

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{pmatrix}.$$

Solution.

1. $\Delta = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} = 2\cdot 1 - 3\cdot(-10) - 4\cdot(-7) = 60 \neq 0,$

2. $\tilde{A} = \begin{pmatrix} 1 & 7 & 17 \\ 10 & 10 & 10 \\ -7 & 11 & 1 \end{pmatrix}$, (see the solution of the example 1.6.1),

3. $$A^{-1} = \frac{1}{\Delta}\tilde{A} = \frac{1}{60}\begin{pmatrix} 1 & 7 & 17 \\ 10 & 10 & 10 \\ -7 & 11 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{60} & \frac{7}{60} & \frac{17}{60} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{7}{60} & \frac{11}{60} & \frac{1}{60} \end{pmatrix}.$$

Definition. A matrix is said to be a **nonsingular** matrix if its determinant is not equal to zero.

§ 1.7. Solution of System of *n* Linear Equation in *n* Unknowns

*Definition.*The system of equation of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22\,2}x + \ldots + a_{2n}x_2 = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases} \qquad (1.7.1)$$

is called the system of n linear equation with n unknowns.

Denoting matrix of the coefficients as A, the column of unknowns as X and the column of the constant terms as B it is possible to rewrite the system (1.7.1) in the **matrix form**

$$AX = B \tag{1.7.2}$$

Here $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$.

Assume that the matrix A is nonsingular matrix. It means that A^{-1} exists. In this case we have

$$AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow \left(A^{-1}A\right)X = A^{-1}B \Rightarrow IX = A^{-1}B,$$

or

$$X = A^{-1}B \tag{1.7.3}$$

Conclusion. If the determinant of the system (1.7.1) does not equal zero then this system has a **unique solution** defined by the formula (1.7.3).

Example 1. 8.1. Solve a system of equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + 3x_3 = 2 \\ x_1 + x_2 + x_3 = 2 \end{cases}$$

The matrix form of the given system is $AX = B$, where

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.$$

$$\det A = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0.$$

Thus we are able to use the formula (1.7.3):

$$X = A^{-1}B.$$

As we know

$$A^{-1} = \frac{1}{\det A}\tilde{A}.$$

Let us find $\tilde{A}$:

$$A_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2, \quad A_{21} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1, \quad A_{31} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$$

$$A_{12} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \quad A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1,$$

$$A_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$\tilde{A} = \begin{pmatrix} -2 & -1 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix}.$$

As $\det A = 1$

$$A^{-1} = \tilde{A} = \begin{pmatrix} -2 & -1 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix}.$$

Then

$$X = A^{-1}B = \begin{pmatrix} -2 & -1 & 5 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -6-2+10 \\ 3+0-2 \\ 3+2-6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Thus the solution of the given system is:

$$\begin{cases} x_1 = 2, \\ x_2 = 1, \\ x_3 = -1. \end{cases}$$

It is possible to prove **Cramer's Rule**:
Let

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22\,2}x + \ldots + a_{2n}x_2 = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

be a system of *n* equations in *n* variables. The solution of the system is given by $(x_1, x_2, x_3, \ldots, x_n)$, where

$$x_1 = \frac{\Delta_1}{\Delta}, \quad \mathrm{x}_2 = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad x_i = \frac{\Delta_i}{\Delta}, \quad \ldots, \quad x_n = \frac{\Delta_n}{\Delta} \qquad (1.7.4)$$

and Δ is the determinant of the coefficient matrix, $\Delta \neq 0$. Δ_i is the determinant formed by replacing the *i*th column of the coefficient matrix with the column of constants $b_1, b_2, b_3, \ldots, b_n$.

Example 1.8.2. Using Cramer's rule solve a system of equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 2x_1 + x_2 + 3x_3 = 2 \\ x_1 + x_2 + x_3 = 2 \end{cases}$$

Solution.

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0.$$

We are able to use formulae (1.7.4):

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta} \ ,$$

where

$$\Delta_1 = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 3\cdot(-2) - 2\cdot(-4) = 2 \Rightarrow x_1 = \frac{2}{1} \Rightarrow x_1 = 2,$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = -4 + 3 + 2 = 1 \Rightarrow x_2 = 1,$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -4 + 3 = -1 \Rightarrow x_3 = -1.$$

The answer: $x_1 = 2, \ x_2 = 1, \ x_3 = -1.$

• *Remember*:

A system of n linear equations in *n* variables has a unique solution if and only if the determinant of the coefficient matrix is not zero.

§ 1.8. The Rank of Matrix

Let be given an arbitrary matrix A dimension of which is $[m\text{x}n]$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Let us strike out k rows and k columns in this matrix.Then *e*lements a_{ij} found at the intersection of these rows and columns form the matrix of order k.

Definition. Determinant of this matrix is said to be **minor of the *k* order of the matrix *A*.**

Definition. The highest order of the minor of matrix A different from zero is called the **rank** of this matrix and denoted ***r(A).***

Definition. Matrix A is **equivalent** to matrix B if their ranks are equal:

$$A \sim B \text{ , if and only if } r(A) = r(B) \qquad (1.8.1)$$

Elementary Operations on Matrices

1. Deleting any row (column) all elements of which are zeros.
2. Interchanging any two rows (columns).
3. Multiplying all elements in a row (column) by the same nonzero number.
4. Replacing a row (column) by the linear combination of this row (column) and any other row (column).

You can prove the Gauss theorem:
Theorem. The elementary operations do not change a rank of a matrix.
Example. Using the elementary operations find the rank of the matrix A.

$$A = \begin{pmatrix} 1 & 3 & 2 & 3 & -1 & 5 \\ 2 & 2 & 4 & 0 & 2 & 3 \\ 3 & 5 & 6 & 3 & 1 & 8 \\ 4 & 8 & 8 & 6 & 0 & 13 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_4 \end{matrix} \sim \begin{pmatrix} 1 & 3 & 2 & 3 & -1 & 5 \\ 0 & -4 & 0 & -6 & 4 & -7 \\ 0 & -4 & 0 & -6 & 4 & -7 \\ 0 & -4 & 0 & -6 & 4 & -7 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 3 & 2 & 3 & -1 & 5 \\ 0 & -4 & 0 & -6 & 4 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 2 & 3 & -1 & 5 \\ 0 & -4 & 0 & -6 & 4 & -7 \end{pmatrix}$$

$$M_2 = \begin{vmatrix} 1 & 3 \\ 0 & -4 \end{vmatrix} = -4 \neq 0 \Rightarrow r(A) = 2.$$

Remark.

*T*o find the rank of a matrix reduce it to a triangular form. A number of nonzero rows is the rank of the given matrix.

§ 1.9. System of Linear Equations in the General Case

Let be given *m* equations in *n* variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases} \tag{1.9.1}$$

Then the matrix of the system (1.9.1) is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The matrix *B* is called the **augmented** matrix of the system (1.9.1):

$$B = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

Definition. A system (1.9.1) having at least one solution is called a **compatible** system.

Theorem. A system (1.9.1) is compatible if and only if the rank of matrix A equals the rank of matrix B.

There is the Gauss Method. The goal of this method is to rewrite an augmented matrix in triangular form. After that it is able to answer the next questions:

1. Is the given linear system compatible or not?

2. How many solutions has this system?

If the system is compatible you can find its solution.

We will now demonstrate how to solve a system of two equations in two variables by the Gauss method. Consider the system of equations

$$\begin{cases} 2x_1 + 5x_2 = -1 \\ 3x_1 - 2x_2 = \ \ 8 \end{cases}$$

The augmented matrix for this system is

$$B = \left(\begin{array}{cc|c} 2 & 5 & -1 \\ 3 & -2 & 8 \end{array}\right) \begin{matrix} \\ 2R_2 - 3R_1 \end{matrix} \sim \left(\begin{array}{cc|c} 2 & 5 & -1 \\ 0 & -19 & 19 \end{array}\right).$$

The system of equation written from the triangular matrix is:

$$\begin{cases} 2x_1 + 5x_2 = -1 \\ \quad\ \ -19x_2 = 19 \end{cases}$$

1) $-19x_2 = 19 \Rightarrow x_2 = -1;$

2) $2x_1 + 5(-1) = -1 \Rightarrow 2x_1 = 4 \Rightarrow x_1 = 2.$

The solution of the given system is (2, -1).

Example 1.9.1 .Solve by using the Gauss method

$$\begin{cases} 2x_1 + x_2 + \quad x_3 + 2x_4 = \ 8 \\ x_1 \text{ - } \ x_2 + 3x_3 + x_4 = 10 \\ x_1 + x_2 \qquad\quad + \quad x_4 = \ 5 \end{cases}.$$

Solution.

Reduce the augmented matrix to triangular form:

$$B = \begin{pmatrix} 2 & 1 & 1 & 2 & | & 8 \\ 1 & -1 & 3 & 1 & | & 10 \\ 1 & 1 & 0 & 1 & | & 5 \end{pmatrix} R_3 \to R_1 \quad \sim \begin{pmatrix} 1 & 1 & 0 & 1 & | & 5 \\ 2 & 1 & 1 & 2 & | & 8 \\ 1 & -1 & 3 & 1 & | & 10 \end{pmatrix} \begin{matrix} R_2 - 2R_1 \sim \\ R_2 - R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 0 & 1 & | & 5 \\ 0 & -1 & 1 & 0 & | & -2 \\ 0 & -2 & 3 & 0 & | & 5 \end{pmatrix} R_3 - 2R_2 \quad \sim \begin{pmatrix} 1 & 1 & 0 & 1 & | & 5 \\ 0 & -1 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & 0 & | & 9 \end{pmatrix}$$

As we can see $r(A) = r(B) = 3$. But $n = 4 > r = 3$. It means that the general solution depends on $n - r$ arbitrary constants. Here is one constant in this case. Let $x_4 = C$, then the equivalent system is

$$\begin{cases} x_1 + x_2 \qquad = 5 - C \\ \quad - x_2 + x_3 = -2 \\ \qquad\qquad x_3 = 9 \end{cases}.$$

Solving this system we have

$$\begin{cases} x_3 = 9, \\ -x_2 + 9 = -2 \Rightarrow x_2 = 11, \\ x_1 + 11 = 5 - C \Rightarrow x_1 = -6 - C \end{cases}.$$

Check up this solution of the given system.

The answer: $(-6-C,\ 11,\ 9)$.

§ 1.10. Homogeneous System of Equations

Definition. A linear system of equations for which the constant term is zero for all equations is called a **homogeneous** system of equations.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases} \qquad (1.10.1)$$

System (1.10.1) is compatible as the solution $(0.0.\ldots.0)$ is always the solution of homogeneous system. If a homogeneous system has non-zero solutions you can find them using the Gauss method.

Example 1.10.1. Solve the homogeneous systems of equations.

$$\text{a)}\begin{cases} x_1 - x_2 + 5x_3 = 0, \\ 2x_1 + x_2 - 4x_3 = 0, \\ -x_1 + 2x_2 - 5x_3 = 0, \end{cases} \qquad \text{b)}\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 0, \\ -x_1 + 3x_2 - 2x_3 - x_4 = 0, \\ x_1 - 3x_2 + x_3 + x_4 = 0, \\ 2x_1 - x_2 + x_3 + 2x_4 = 0. \end{cases}$$

Solution. Solve the system reducing the matrix of the system to a triangular form.

$$\text{a)}\ A = \begin{pmatrix} 1 & -1 & 5 \\ 2 & 1 & -4 \\ -1 & 2 & -5 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{pmatrix} 1 & -1 & 5 \\ 0 & 3 & -14 \\ 0 & 3 & -10 \end{pmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & -1 & 5 \\ 0 & 3 & -14 \\ 0 & 0 & 4 \end{pmatrix}.$$

Thus the equivalent system has the unique trivial solution:

$$\begin{cases} x_1 - x_2 + 5x_3 = 0 \\ 3x_2 - 14x_3 = 0 \\ 4x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}.$$

b) $$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 3 & -2 & -1 \\ 1 & -3 & 1 & 1 \\ 2 & -1 & 1 & 2 \end{pmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 5 & -3 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & -5 & 3 & 0 \end{pmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + R_2 \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As we can see $r(A) = 3$. But $n = 4 > r = 3$. It means that the general solution depends on $n - r = 4 - 3 = 1$ arbitrary constants.

Let us write out the system, which is equivalent to the given one:

$$\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 0, \\ 5x_2 - 3x_3 = 0, \\ -x_3 = 0. \end{cases}$$

Let $x_4 = C$, then the solution of the system is $X = \begin{pmatrix} -C \\ 0 \\ 0 \\ C \end{pmatrix}$.

As the given system can be written in the form

$$A \cdot X = 0, \text{ where } A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 3 & -2 & -1 \\ 1 & -3 & 1 & 1 \\ 2 & -1 & 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \; 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

it is possible to check the solution in matrix form:

$$A \cdot X = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 3 & -2 & -1 \\ 1 & -3 & 1 & 1 \\ 2 & -1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -C \\ 0 \\ 0 \\ C \end{pmatrix} = \begin{pmatrix} -C+C \\ C-C \\ -C+C \\ -2C+2C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The answer: $X = \begin{pmatrix} -C \\ 0 \\ 0 \\ C \end{pmatrix}$.

§ 1.11. Miscellaneous Problems

1. Let the matrix $\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 3 \\ 4 & 3 & -5 & 0 \\ 3 & 2 & 0 & -5 \end{pmatrix}$ be given. Calculate *detA*=$|A|$, using the properties of determinates.

2. Calculate minor M_{13} and cofactor A_{22} of the matrix *A* from the previous problem.

3. Find the matrix $D = 2A - 3B$ if

$$A=\begin{pmatrix}1 & 2 & -3 & 4\\ 5 & -1 & 2 & 0\end{pmatrix},\ B=\begin{pmatrix}7 & -2 & 0 & 3\\ 1 & 5 & 2 & -2\end{pmatrix}.$$

4. Find the product of matrices

a) $\begin{pmatrix}2 & 1 & -3\\ 5 & -7 & 4\\ 13 & 0 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & 2 & -5\\ 1 & 0 & 5\\ 0 & 7 & 1\end{pmatrix}$;

b) $\begin{pmatrix}1 & 3\\ 2 & -1\\ 4 & 0\end{pmatrix}\cdot\begin{pmatrix}-2 & -1\\ 4 & 5\end{pmatrix}$.

5*. There are two linear transformations

$$\begin{cases} y_1=-x_1-x_2-x_3\\ y_2=-x_1+4x_2+7x_3\\ y_3=8x_1+x_2-x_3\end{cases},\ \text{and}\ \begin{cases} z_1=9y_1+3y_2+5y_3\\ z_2=2y_1+3y_3\\ z_3=y_2+y_3\end{cases}.$$

Find the transformation *Z* through *X*.

6. Find the inverse matrix A^{-1} of the matrix $A=\begin{pmatrix}2 & -1 & 0\\ 5 & 3 & -6\\ 1 & -2 & 3\end{pmatrix}$. Check the equality

$AA^{-1}=A^{-1}A=I.$

7. Solve the system of linear equations using a)the matrix method, b) Cramer's rule, c) Gauss method :

$$\begin{cases} 7x_1-5x_2=-1\\ 2x_1+x_2-15x_3=9.\\ x_1+2x_2-9x_3=2\end{cases}$$

8. Find the rank of the matrices

a) $A = \begin{pmatrix} 1 & 9 & 8 & -2 \\ 1 & 2 & 3 & -2 \\ 2 & -3 & 1 & -4 \end{pmatrix}$ b) $B = \begin{pmatrix} 4 & 2 & -1 \\ 1 & 0 & -7 \\ 3 & 1 & -5 \\ 2 & -1 & -3 \end{pmatrix}$

9. Solve the system of linear equations

$$\begin{cases} x_1 + 2x_2 + 3x_3 - x_4 = 1 \\ 3x_1 + 2x_2 + x_3 - x_4 = 1 \\ 2x_1 + 3x_2 + x_3 + x_4 = 1 \\ 5x_1 + 5x_2 + 2x_3 \quad = 2 \end{cases}$$

10. Solve the homogenous system

$$\begin{cases} x_1 - x_2 + 2x_3 + 4x_4 = 0 \\ 4x_1 \quad + 4x_3 + 9x_4 = 0 \\ 3x_1 + x_2 + 2x_3 + 5x_4 = 0 \\ x_1 + 3x_2 - 2x_3 - 3x_4 = 0 \end{cases}$$

11*. Let the function $f(t) = -t^2 + 3t + 4$ be given. Find $f(A)$, if

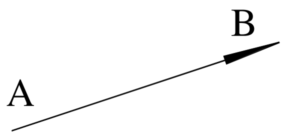
$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$

12** Find characteristic numbers and characteristic vectors of the matrix

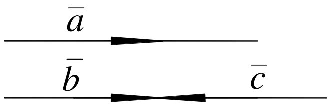
$$A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$$

PART II
ALGEBRA OF VEKTORS

§ 2.1. Definitions

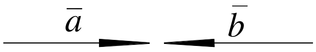


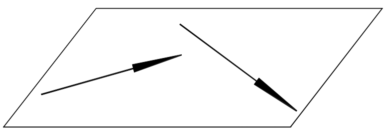
<u>*Definition 2.1.1.*</u> **Vector** is a directed line segment.
A is the initial point,
B is the terminal (end) point.
$\overline{AB} = \overline{a}$.



*Definition 2.1.2.*_Vectors lying on the parallel straight lines or on the one and the same straight line are called **collinear** (*коллинеарные, параллельные).*

$\vec{a} \mid\mid \overline{b} \mid\mid \overline{c}$

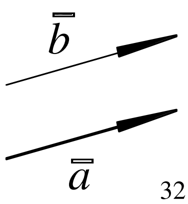




*Definition 2.1.3.*_Vectors lying on the straight lines parallel one and the same plane said to be **coplanar**.
$\overline{a}, \overline{b}, \overline{c}, \overline{d}$ are the coplanar vectors.

Definition 2.1.4. The length of a vector $\overline{a}$ is called its **modulus** $|\overline{a}|$.

*Definition 2.1.5.*_The vectors $\overline{a}$ and $\overline{b}$ are called equal if they are collinear and have the same length and direction.

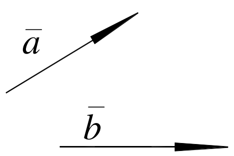


$$\begin{cases} |a| = |b|, \\ \bar{a} \,||\, \bar{b}, \\ \bar{a} \uparrow\uparrow \bar{b} \end{cases} \qquad (2.1.1)$$

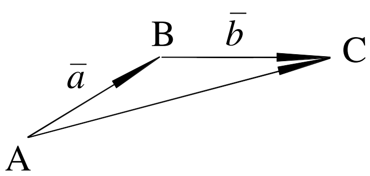
§ 2.2. The Linear Operations

Triangle's rule (the definition of the sum of two vectors).
Let $\bar{a}$ and $\bar{b}$ be the given vectors. Draw the vector $\bar{b}$ from the terminal point of the vector $\bar{a}$**.** The sum $\bar{a} + \bar{b}$ is the vector extending from the initial point of the vector $\bar{a}$ to the terminal point of the vector $\bar{b}$**.**

Example 2.2.1. Find the sum of the given vectors $\bar{a}$ and $\bar{b}$.
Solution. Let us take any point A in the plane. Draw the vector $\overline{AB} = \bar{a}$.



Then draw the vector $\overline{BC} = \bar{b}$. So by the triangle's rule the vector AC is equal to the sum of the given vectors $\bar{a}$ and $\bar{b}$.

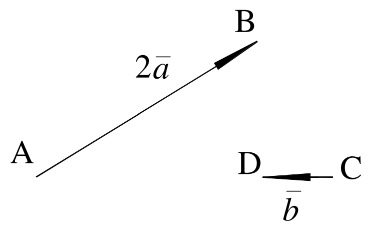


$$\bar{a} + \bar{b} = \overline{AC}$$

<u>*Definition 2.2.1.*</u> The product of a vector $\bar{a}$ by a scalar (number) $\lambda \neq 0$ is the vector $\bar{b}$ such that

$$\begin{aligned} &|\bar{b}| = |\lambda| \cdot |\bar{a}| \\ &\bar{a} \uparrow\uparrow \bar{b}, \text{ if } \lambda > 0, \\ &\bar{a} \uparrow\downarrow \bar{b}, \text{ if } \lambda < 0. \end{aligned} \qquad (2.2.1)$$

Example 2.2.2. Draw the vectors $2\overline{a}$ and $-\frac{1}{3}\overline{b}$. Take the vectors $\overline{a}$ and $\overline{b}$ from the *Example 2.2.1.*

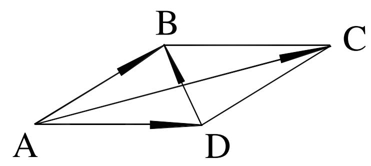


Solution.

By the definition 2.1 the vector $\overline{AB} = 2\overline{a}$, and $\overline{CD} = -\frac{1}{3}\overline{b}$.

Parallelogram's Rule.

Prove that

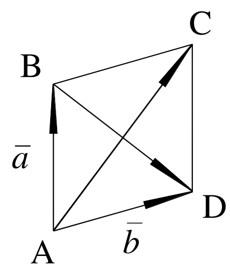


1) $\overline{a} + \overline{b} = \overline{AC}$ is the sum of vectors $\overline{a} = \overline{AB}$ and $\overline{b} = \overline{AD}$,

2) $\overline{a} - \overline{b} = \overline{DB}$ - is the difference of the vectors $\overline{a}$ and $\overline{b}$.

Example 2.2.3.

Let the vectors $\overline{a}$ and $\overline{b}$ be given. Find $\overline{a} + \overline{b}$ and $\overline{b} - \overline{a}$ in the one and the same drawing.
Solution. To solve this problem construct a parallelogram ABCD with the sides AB = $|\overline{a}|$ and AD = $|\overline{b}|$, then the diagonal $\overline{AC}$ is the sum of these vectors, and the other diagonal directed to the vector- minuend is the difference of them

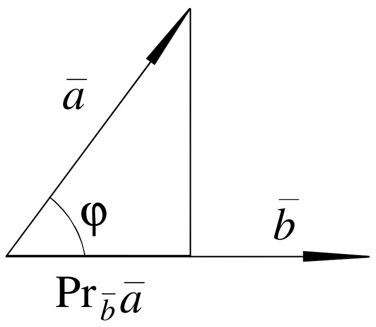


$$\overline{AC} = \overline{a} + \overline{b},$$

$$\overline{BD} = a - \overline{b}.$$

§1.3. The Scalar Product of Two Vectors

Definition 2.3.1. The scalar product (dot product) of vectors $\overline{a}$ and $\overline{b}$ is the number equal to the product of the moduli of these vectors and the cosine of the angle φ between them.



$$\overline{a} \cdot \overline{b} = (\overline{a}, \overline{b}) = |\overline{a}| \cdot |\overline{b}| \cos\varphi = |\overline{b}| Pr_{\overline{b}}\, \overline{a},$$

where $Pr_{\overline{b}}\, \overline{a}$ is the **vector projection** of $\overline{a}$ onto $\overline{b}$.

$$Pr_{\overline{b}}\, \overline{a} = |\overline{a}| \cos\varphi \qquad (1.3.1a)$$

$$Pr_{\overline{b}}\, \overline{a} = \frac{(\overline{a}, \overline{b})}{|\overline{b}|} \qquad (1.3.1b)$$

$$\cos\varphi = \frac{(\overline{a}, \overline{b})}{|\overline{a}| \cdot |\overline{b}|} \qquad (1.3.2)$$

§ 2.4. The Properties of the Scalar Product

1. $(\overline{a}, \overline{b}) = (\overline{b}, \overline{a})$ – commutative law of the scalar product.

2. $(\lambda\overline{a}, \overline{b}) = \lambda(\overline{a}, \overline{b})$ – associative law with respect to multiplication by a number. (The scalar λ can be taken out of the scalar product).

3. $(\overline{a}, \overline{b} + \overline{c}) = (\overline{a}, b) + (\overline{a}, \overline{c})$ – distributive law with respect to addition.

4. The scalar product of two non-zero vectors equals zero if and only if they are perpendicular (orthogonal).

The proof.

$$(\overline{a}, \overline{b}) = 0 \Leftrightarrow |\overline{a}| \cdot |\overline{b}| \cos\varphi = 0 \Leftrightarrow \cos\varphi = 0 \Leftrightarrow \varphi = 90^0.$$

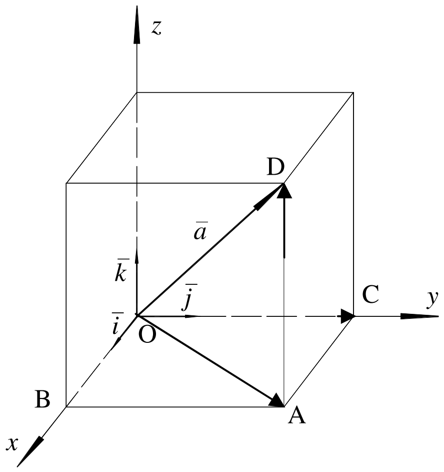
We'll use this property in such a way:

$$\overline{a} \perp \overline{b} \Leftrightarrow (\overline{a}, \overline{b}) = 0 \qquad (2.4.1)$$

5. The scalar product of a vector with itself is equal to the square of its modulus.
$(\overline{a}, \overline{a}) = |\overline{a}|^2$

§2.5. The Coordinates of a Vector

Let $\bar{i}, \bar{j}, \bar{k}$ be the unit and orthogonal vectors giving the direction of x-axis, y-axis and z-axis accordingly.



$|\bar{i}| = |\bar{j}| = |\bar{k}| = 1,\ \bar{i} \perp \bar{j} \perp \bar{k}.$

$\bar{a} = \overline{OD} = \overline{OA} + \overline{AD} = \overline{OB} + \overline{OC} + \overline{AD} =$

$= x\,\bar{i} + y\,\bar{j} + z\,\bar{k},$

where

$x = \Pr_{\bar{i}}\bar{a} = a_x$,

$y = \Pr_{\bar{j}}\bar{a} = a_y$,

$z = \Pr_{\bar{k}}\bar{a} = a_z$

are the projections of the vector $\bar{a}$ onto the vectors $\bar{i}, \bar{j}$ *and* $\bar{k}$ accordingly. These numbers are called the **coordinates** of the vector $\bar{a}$.

$\bar{a} = (a_x, a_y, a_z)$ is the **coordinate form** of the vector $\bar{a}$**;**

$\bar{a} = a_x\bar{i} + \bar{a}_y\bar{j} + a_z\bar{k}$ – **the vector form** of the vector $\bar{a}$, or **the expansion of the vector $\bar{a}$ through the base $\bar{i}, \bar{j}, \bar{k}$.**

1. Let $\bar{a} = (a_x, a_y, a_z)$, $\bar{b} = (b_x, b_y, b_z)$ be given vectors, then

a) $|\bar{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$; (2.5.1)

b) $\lambda\bar{a} = (\lambda\bar{a}_x, \lambda a_y, \lambda a_z)$ (2.5.2)

c) $\bar{a} \pm \bar{b} = (a_x \pm b_x, a_y \pm b_y, a_z \pm b_z)$ (2.5.3)

d) $(\bar{a}, \bar{b}) = a_x b_x + a_y b_y + a_z b_z$ (2.5.4)

e) $\bar{a} \perp \bar{b} \Leftrightarrow a_x b_x + a_y b_y + a_z b_z = 0$ (2.5.5)

f) $\bar{a} \,||\, \bar{b} \Leftrightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$ (2.5.6)

g) $\cos\left(\overline{a} \wedge \overline{b}\right) = \cos\varphi = \dfrac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}}$ (2.5.7)

Example 2.5.1.

Let the vectors $\overline{a}$ = (3; -2; 1) and $\overline{b}$ = (-6; 4; -2) be given. Then

a) $\left|\overline{a}\right| = \sqrt{9+4+1} = \sqrt{14}$;

b) $3\overline{a}$ = (9; - 6; 3);

c) $\overline{a}+\overline{b}$ = (3 – 6; -2 + 4; 1- 2) $\Rightarrow \overline{a}+\overline{b}$ = (-3; 2; -1);

$\overline{a}-\overline{b}$ = (3 – (-6); -2 – 4; 1 – (-2)) $\Rightarrow \overline{a}-\overline{b}$ = (9; - 6; 3);

d, e) $\left(\overline{a},\overline{b}\right)$ = 3 (-6) + (- 2) 4 + 1 (-2) = -18 – 8 – 2 = - 28 $\neq$ 0, so these vectors are not perpendicular;

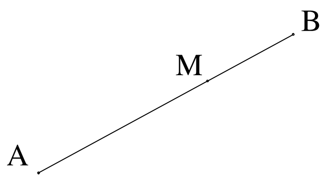
f) $\overline{a} \parallel \overline{b}$, as $\dfrac{3}{-6} = \dfrac{-2}{4} = \dfrac{1}{-2} = -\dfrac{1}{2}$;

g) $\cos\left(\overline{a} \wedge \overline{b}\right) = \dfrac{-28}{\sqrt{14}\sqrt{56}} = \dfrac{-28}{28} = -1 \Rightarrow \varphi = \arccos(-1) = \pi$.

2. Let two points A($x_A; y_A; z_A$) and B(x_B, y_B, z_B) be given, then

a) $\overline{a} = \overline{AB} = \left(x_B - x_A; y_B - y_A; z_B - z_A\right)$ (2.5.7)

b) if *x, y, z* denote the coordinates of the point M dividing the segment AB in the given ratio AM:MB = λ, then



$x = \dfrac{x_A + \lambda x_B}{1+\lambda}$; $y = \dfrac{y_A + \lambda y_B}{1+\lambda}$; $z = \dfrac{z_A + \lambda z_B}{1+\lambda}$; (2.5.8)

c) in particular the coordinates of the **midpoint** of the given segment AB are

$$x = \frac{x_A + x_B}{2};\ y = \frac{y_A + y_B}{2};\ z = \frac{z_A + z_B}{2} \qquad (2.5.9)$$

Example 2.5.2. Find the coordinates of

a) the vector $\overline{AB}$,
b) the midpoint C of the segment AB;
c) the point M such that $AM : MB = 2 : 1$ if $A(3; -5; 2)$, $B(1; 4; -1)$.

Solution.

a) Using the formula (2.5.7) we have $\overline{AB} = (1- 3; 4-(-5); -1-2)$,
so $\overline{AB} = (-2;9;-3)$.

b) By the formula (2.5.9) the coordinates of the midpoint C are

$$x = \frac{3+1}{2} \Rightarrow x = 2;\ y = \frac{-5+4}{2} \Rightarrow y = -\frac{1}{2};\ z = \frac{2-1}{2} \Rightarrow z = \frac{1}{2},$$

so $C(2; -\frac{1}{2}; \frac{1}{2})$.

c) As $AM : MB = 2 : 1$, then $\lambda = 2$. From the formula (2.5.8) we can find the coordinates of the point M:

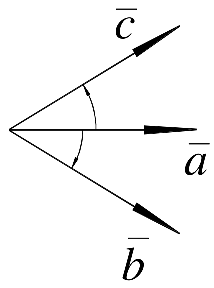
$$x = \frac{3 + 2\cdot 1}{1+2} = \frac{5}{3},\quad y = \frac{-5+8}{3} = 1,\quad z = \frac{2-2}{3} = 0.$$

So $M(5/3; 1; 0)$.

§ 2.6. The Vector Product of two Vectors (the cross product)

Definition 2.6.1. If we indicate the sequence of order of the triple of vectors,then this triple of vectors is called **ordered triple of vectors**.

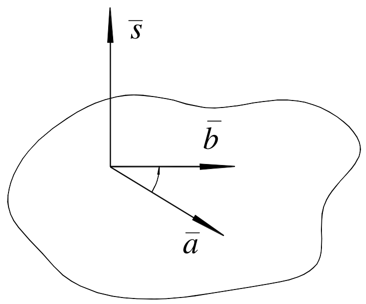
Definition 2.6.2. The ordered triple of vectors is called a right (left) – handed triple if the shortest rotation of the first vector to the second one is observed from the end point of the third vector in the counterclockwise (clockwise).



$\bar a, \bar b, \bar c$ – is the left-handed triple,

$\bar a, \bar c, \bar b$ – is the right-handed triple.

Definition 2.6.3. The vector product (cross product) of two vectors $\bar a$ and $\bar b$ is the vector $\bar s$ such that



1) $|\bar s| = |\bar a||\bar b| \sin\varphi$, where φ is the angle between $\bar a$ and $\bar b$;
2) $\bar s \perp \bar a$, $\bar s \perp \bar b$ (the vector $\bar s$ is orthogonal to both of the vectors $\bar a$ and $\bar b$);
3) $\bar a, \bar b, \bar s$ is the right-handed triple of vectors.

$$\bar s = [\bar a, \bar b] = \bar a \times \bar b$$

§ 2.7. The Coordinate Form of the Vector Product

Let $\bar a = (a_x, a_y, a_z)$ and $\bar b = (b_x, b_y, b_z)$ be given vectors, then the coordinate form of the vector product is

$$[\bar a, \bar b] = \begin{vmatrix} \bar i & \bar j & \bar k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \bar i \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \bar j \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \bar k \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \qquad (2.7.1)$$

Example 2.7..1. Calculate $[\bar a, \bar b]$ if $\bar a = 2\bar i - 7\bar j + 4\bar k$, $\bar b = (1,\text{-}1,0)$.

Solution. As we know the coefficients of $\bar i, \bar j, \bar k$ are the first, second and third coordinates accordingly of the vector $\bar a$. So by the formula (2.7.1) we have

$$[\bar a, \bar b] = \begin{vmatrix} \bar i & \bar j & \bar k \\ 2 & -7 & 4 \\ 1 & -1 & 0 \end{vmatrix} = \bar i \begin{vmatrix} -7 & 4 \\ -1 & 0 \end{vmatrix} - \bar j \begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} + \bar k \begin{vmatrix} 2 & -7 \\ 1 & -1 \end{vmatrix} = -4\bar i + 4\bar j + 5\bar k.$$

§ 2.8. The Properties of the Vector Product

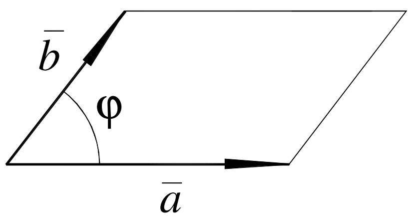
1. The vector product is anticommutative:

$$[\overline{a},\overline{b}] = -[\overline{b},\overline{a}].$$

2. The vector product is associative with respect to multiplication by the scalar: $[\lambda\overline{a},\overline{b}] = [\overline{a},\lambda\overline{b}] = \lambda[\overline{a},\overline{b}]$, a scalar is taken out of the (square) brackets.
3. The vector product is distributive with respect to addition: $[\overline{a}+\overline{b},\overline{c}] = [\overline{a},\overline{c}] + [\overline{b},\overline{c}]$.
4. Let $\overline{a} \neq 0$, $\overline{b} \neq 0$, then $[\overline{a},\overline{b}] = 0$ if and only if these vectors are collinear: $[\overline{a},\overline{b}]=0 \Leftrightarrow \overline{a} \,||\, \overline{b}$.

   In fact,
   $|[\overline{a},\overline{b}]| = 0 \Leftrightarrow |\overline{a}||\overline{b}|\sin\varphi = 0 \Leftrightarrow \sin\varphi = 0 \Leftrightarrow \varphi = 0, \text{ or } \pi \Leftrightarrow \overline{a} \,||\, \overline{b}.$

5. Geometrical property of the vector product.



Let us construct the parallelogram on the given vectors $\overline{a}$ and $\overline{b}$ as the sides. φ is the angle between these vectors ($\varphi = (\overline{a} \wedge \overline{b})$).

As we know the area of a parallelogram is calculated by the formula: $S = |\overline{a}| \cdot |\overline{b}| \sin(\overline{a} \wedge \overline{b})$

But $|\overline{a}|\,|\overline{b}| \sin(\overline{a} \wedge \overline{b}) = |\overline{a}| \cdot |\overline{b}| \sin\varphi = |[\overline{a},\overline{b}]|$. So we have

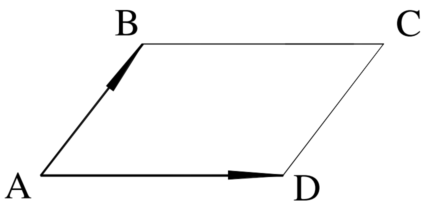
$$S = |[\overline{a},\overline{b}]| \qquad (2.8.1)$$

Geometrically **a magnitude of the vector product is equal to the area of the parallelogram constructed on these vectors.**

Example 2.8.1.

Find the area of the parallelogram *ABCD* if *A*(9; 2;-5), *B*(2;1;1), *D*(9;2;0).

Solution.
First of all let us do a drawing.



By the formula (2.8.1) we have

$$S = |[\overline{AB}, \overline{AD}]|$$

1). $\overline{AB}$ = (-7;-1;6), $\overline{AD}$ = (0;0;5);

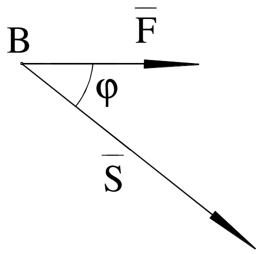
2). $$[\overline{AB}, \overline{AD}] = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -7 & -1 & 6 \\ 0 & 0 & 5 \end{vmatrix} = \bar{i}\cdot(-5) - \bar{j}\cdot(-35)\bar{k}\cdot 0 = -5\bar{i} + 35\bar{j};$$

3). $$|[\overline{AB}, \overline{AD}]| = \sqrt{5^2 + 35^2} = \sqrt{5^2(1+7^2)} = 5\sqrt{50} = 25\sqrt{5}.$$

The answer: $S_{ABCD} = 25\sqrt{2}.$

§2.9. Mechanical Properties of the Scalar and Vector Products

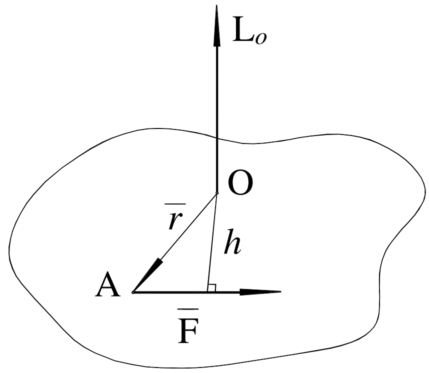
1. $\overline{F}$ is the force acting at the point B, $\overline{S}$ is the displacement , A is the work, done by the force $\overline{F}$ along the displacement $\overline{S}$.



As we know

$$A = |\overline{F}||\overline{S}|\cos\varphi = (\overline{F}, \overline{S}) \qquad (2.9.1)$$

Mechanically: *the scalar product is the work***.**



2. Let O is a pivot. As we know the torque (*момент вращения*) $\overline{L}_o \perp \overline{r}$, $L_o \perp \overline{F}$ and

$$|\overline{L}_o| = |\overline{F}|h = |\overline{F}||\overline{r}|\sin\varphi = |[\overline{r}, \overline{F}]|.$$

Mechanically; *the vector product is the torque of the force :*

$$\left[\overline{r},\overline{F}\right]=\overline{L}_o \qquad (2.9.2)$$

Example. Find the torque $\overline{L}_o$ of the force $\overline{F}_A$ if $A(3;-1;7)$, $O(0;0;0)$,

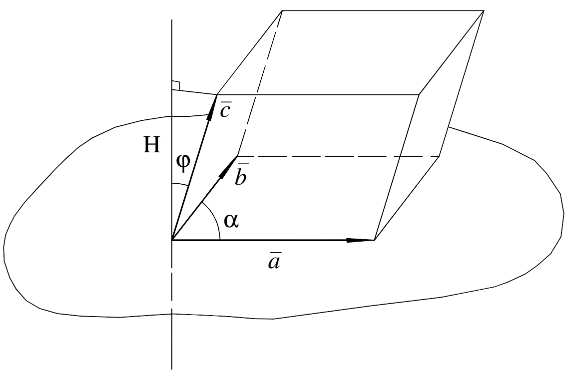
$\overline{F}_A = (2; 8; -2)$.

Solution. Use the formula (2.9.2) where $\overline{r}=\overline{OA}=$ (3;-1;7), $\overline{F}_A=$ (2;8;-2). Thus

$$\overline{L}_o=\left[\overline{OA},\overline{F}_A\right]=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 3 & -1 & 7\\ 2 & 8 & -2\end{vmatrix}=-54\overline{i}+20\overline{j}+26\overline{k}$$

§ 2.10. The Triple Scalar Product

Definition. The product $\left(\overline{a},\overline{b},\overline{c}\right)=\left(\left[\overline{a},\overline{b}\right],\overline{c}\right)=\left(\overline{a},\left[\overline{b},\overline{c}\right]\right)$ is called the **triple scalar product**.



$$\left[\overline{a},\overline{b}\right]=\overline{S},$$

$H=\left|\overline{c}\right|\cos\varphi = \pm$ altitude of a box, H is a height,

$|\overline{S}|$ is the area of the base **,** so

$|\overline{S}| = |\overline{a}|\,|\overline{b}|\sin\alpha$,

then the volume of the parallelepiped is $V=\left|\overline{S}\right|H=\left|\overline{a}\right|\left|\overline{b}\right|\sin\alpha\left|\overline{c}\right|\cos\varphi=\pm\left|\overline{S}\right|\left|\overline{c}\right|\cos\varphi=\pm\left(\overline{S},\overline{c}\right)=\pm\left(\overline{a},\overline{b},\overline{c}\right)$.

The geometrical meaning of the triple scalar product *is that the triple scalar product equals the volume of the parallelepiped determined by these vectors taking with the sign "+" if the triple is the right handed and with the sign " - " if it is the left handed triple***.**

We'll use it as

$$V=\left|\left(\overline{a},\overline{b},\overline{c}\right)\right| \qquad (2.10.1)$$

§ 2.11. The Coordinate Form of the Triple Scalar Product

Let $\bar{a}=(a_x,a_y,a_z)$, $\bar{b}=(b_x,b_y,b_z)$, $\bar{c}=(c_x,c_y,c_z)$, then

$$(\bar{a},\bar{b},\bar{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad (2.11.1)$$

Prove formula (2.11.1) by steps:

1). $(\bar{a},\bar{b},\bar{c})=(\bar{a},[\bar{b},\bar{c}])$;

2). $[\bar{b}.,\bar{c}]=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=\bar{s}$;

3). $(\bar{a},\bar{s})=a_x s_x+a_y s_y+a_z s_z$.

Example.

a) Calculate the volume of the parallelepiped constructed on the vectors $\bar{a}=2\bar{i}+5\bar{k}$, $\bar{b}=(1;3;-5)$ and $\bar{c}=\overline{AB}$, where $A(-3;5;7)$, $B(2;9;-5)$.

b) Is the triple of the vectors $\bar{a},\bar{b},\bar{c}$ the right-handed or the left-handed triple?

Solution.

a) Use the formula (2.11.1), where $\bar{a}=(2;0;5)$, $\bar{b}=(1;3;-5)$, $\bar{c}=(5,4,12)$. So we have

$$(\bar{a},\bar{b},\bar{c})=\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & -5 \\ 5 & 4 & -12 \end{vmatrix}\begin{matrix} 2 & 0 \\ 1 & 3 \\ 5 & 4 \end{matrix} = 72+20+0-75+40-0=-87.$$

Thus V = |-87| = 87.

b) $\bar{a},\bar{b},\bar{c}$ is the left-handed triple as $(\bar{a},\bar{b},\bar{c})=-87<0$.

§ 2.12. Properties of the Triple Scalar Product

Prove that:

1. $(\overline{a},\overline{b},\overline{c}) = -(\overline{b},\overline{a},\overline{c})$
2. $(\lambda\overline{a},\overline{b},\overline{c}) = \lambda(\overline{a},\overline{b},\overline{c})$
3. $(\overline{a}_1+\overline{a}_2,\overline{b},\overline{c}) = (\overline{a}_1,\overline{b},\overline{c}) + (\overline{a}_2,\overline{b},\overline{c})$
4. Let $\overline{a}\neq 0, \overline{\mathrm{b}}\neq 0, \overline{\mathrm{c}}\neq 0$, then

$$(\overline{a},\overline{b},\overline{c}) = 0 \Leftrightarrow \overline{a},\overline{b},\overline{c} \text{ are coplanar} \qquad (2.12.1)$$

§ 2.13. The Questions for the Test Paper

A. How to find

1) the sum of the given vectors,
2) the difference of the given vectors,
3) the product of a vector and a scalar,

if the vectors are given

a) as directed segments;
b) in the coordinate form?

B. How to find the coordinates of

1) the vector $\overline{AB}$ and its length $|\overline{AB}|$,
2) the midpoint of the segment AB,
3) the point dividing the segment AB in the given ratio λ if the coordinates of the points A and B are given;
4) linear combination $\alpha\overline{a}+\beta\overline{b}+\gamma\overline{c}$, if the coordinates of the vectors $\overline{a},\overline{b},\overline{c}$ are given;
5) the direction cosines of the vector $\overline{a}$, if its coordinates are known?

C. How to find

a) the projection of the vector $\overline{a}$ in the direction of the vector $\overline{b}$, if their coordinates are given?
b) the work done by the force $\overline{F}=\overline{a}$ along a displacement $\overline{AB}=\overline{b}$, if the coordinates of these vectors are known?

D. How to find

a) the interior (exterior) angle of a triangle ABC,
b) the area of a triangle ABC (parallelogram $ABCD$), if the vertices of the figure are known?

E. How to find

a) the vector that is perpendicular to both of the vectors $\overline{a}$ and $\overline{b}$,
b) the torque $\overline{L}_A$ of the force $\overline{F} = \overline{CD}$, if the coordinates of the points *A, C, D* are known?

F. How to find

a) the triple scalar product of the vectors $\overline{a}, \overline{b}, \overline{c}$?
b) the volume of the tetrahedron (parallelepiped), defined of the vectors $\overline{a}, \overline{b}, \overline{c}$, knowing their coordinates?

G. How to define, is the given triple of vectors $\overline{a}, \overline{b}, \overline{c}$ the left-handed or the right-handed triple?

H. How to verify whether the given vectors are

a) perpendicular (orthogonal)?
b) parallel (collinear)?
c) coplanar?

§ 2.14. Miscellaneous Problems

In the exercises 1 – 5 express each of the vectors in the vector form : $x\overline{i} + y\overline{j} + z\overline{k}$ **,** where *x*, *y* and *z* are the coordinates of a required vector.

1. $\overline{P_1P_2}$, where $P_1(1; 3; -1)$, $P_2(2; -1; 0)$.

2. $\overline{OP}$ if *O* is the origin (*начало координат*) and P is the midpoint of the segment P_1P_2 joining P_1(2;-1;3) and P_2(-4;3;5).

3. The vector from the point A(2;3;-7) to the origin.

4. The sum of the vectors $\overline{AB}$ and $\overline{CD}$, where A(1;-1;2), B(2;0;3), C(-1;3;0) and D(-2;2;4).

5. A unit vector of the same direction as the vector $3\bar{i}-4\bar{j}$.

6. Suppose it is known that $(\bar{a},\bar{b}_1)=(\bar{a},\bar{b}_2)$ and $\bar{a}\neq 0$.
Is it permissible to cancel $\bar{a}$ from both sides of the equality?

7. Find the angle $\angle ABC$ of the triangle ΔABC whose vertices are the points A(-1;0;2), B(2;1;-1) and C(1;-2;2). Construct this triangle in the Cartesian system of coordinates.

8. Find the projection of the vector $\bar{b}$ in the direction of the vector $\bar{a}$ if $\bar{a}=\bar{i}-2\bar{j}-2\bar{k}, \bar{b}=(6;3;2)$. What does the sign of the projection mean?

9. Find α if $\bar{a}$ = (2; α; -1), $\bar{b}$ = (3; 4; 3α) and $\bar{a} \,||\, \bar{b}$.

10. Find α and β if $\bar{a}=2\bar{i}+\alpha\,\bar{j}-\bar{k}$, $\bar{b}$ = (3; 4; β) and $\bar{a} \,||\, \bar{b}$.

11. Evaluate the given third order determinant $\begin{vmatrix} 2 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix}$, expanding

a) by the first row;
b) by the second column.

12. Evaluate the given third order determinant, and find M_{22}, A_{13}.

$$|A| = \begin{vmatrix} 3 & 4 & 7 \\ 0 & -2 & 5 \\ 0 & 0 & 5 \end{vmatrix}$$

13. Find the work done by the force $\bar{F}$ along the path $\overline{AB}$ if
$\bar{F}=2\bar{i}-3\bar{j}+\bar{k}$, A(3; 1; 0), B(1; 4; 7).

14. Find the area of the ΔABC given in the Exercise 7.

15. Find the torque $\bar{L}_O$ of the force $\bar{F}=\overline{CD}$, if A(2; 1; 3), C(4; -1; 2), D(0;2;-3).

16. Find the direction cosines cosα, cosβ, cosγ of the vector $\bar{a}=3\bar{i}+4\bar{j}-5\bar{k}$,
knowing that cosα = cos($\bar{i}$ ^ $\bar{a}$), cosβ = cos ($\bar{j}$ ^ $\bar{a}$), cosγ = cos($\bar{k}$ ^ $\bar{a}$).

17. Let $A(-1;5;0)$, $B(2;4;3)$, $C(3;5;-2)$, $D(-1;-2;0)$ be given points. Evaluate

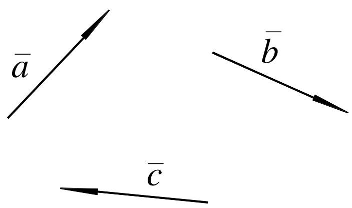
a) the Triple Scalar Product of the vectors $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$;

b) the volume of the tetrahedron $ABCD$,

c) α such that the vectors $\overline{AB}, \overline{AC}$ and $\overline{a} = (1;-2;\alpha)$ be coplanar,

d) β such that the triple $\overline{AC}, \overline{AD}$ and $\overline{b} = (\beta;1;0)$ be left-handed.

18. Let the vectors $\vec{a}, \overline{b}, \overline{c}$ be given. Construct the vectors



1) $\overline{a}+\overline{b}$ and $\overline{b}-\overline{a}$;

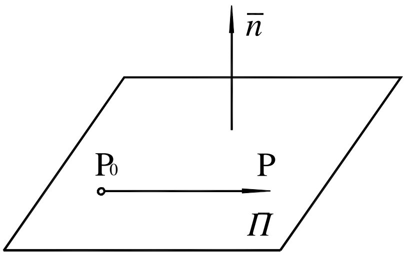
2) $2\overline{a}$, $-\frac{1}{2}\overline{b}$, $3\overline{c}$;

2) $\overline{a}+\overline{b}+\overline{c}$, $-\overline{a}+\overline{b}-\overline{c}$.

PART III. ANALYTIC GEOMETRY

A. PLANES AND LINES IN SPACE

§ 3A.1. The Equation for the Plane Through the Given Point Normal to $\bar{n} = (A, B, C)$



Suppose π is a plane in space that passes through a point $P_0(x_0, y_0, z_0)$ and is normal (perpendicular) to the nonzero vector $\bar{n} = (A, B, C)$, which is called a normal or **a normal vector**.

If a point $P(x, y, z)$ lies on the plane π, then $\bar{n} \perp \overline{P_0 P} \Leftrightarrow (\bar{n}, \overline{P_0 P}) = 0 \Leftrightarrow$

$$\begin{bmatrix} \bar{n} = (A, B, C) \\ \overline{P_0 P} = (x - x_0, y - y_0, z - z_0) \end{bmatrix} \Leftrightarrow$$

$$\underline{\underline{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}} \qquad (3A.1.1)$$

When rearranged, this becomes

$$Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0, \text{ or}$$

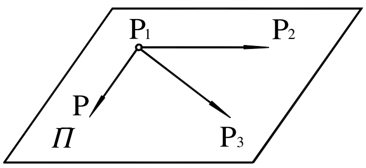
$$\underline{Ax + By + Cz + D = 0}, \qquad (3A.1.2)$$

where $D = -(Ax_0 + By_0 + Cz_0)$.

The equation (3A.1.2) is called the **standard equation of a plane.**

§ 3A.2. The Equation for the Plane through Three Given Points

Let there be given three points not lying on a straight line: $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$. It is required to write the equation of a plane passing through these three points.



Let us take a point $P(x, y, z)$ on the plane π and construct the vectors

$\overline{P_1 P} = (x - x_1, y - y_1, z - z_1)$,

$$\overline{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1),$$

$$\overline{P_1P_3} = (x_3 - x_1, y_3 - y_1, z_3 - z_1).$$

These vectors lie on the plane π, so they are coplanar and therefore their triple scalar product is zero. That is ($\overline{P_1P}$, $\overline{P_1P_2}$, $\overline{P_1P_3}$) = 0, or

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \qquad (3A.2.1)$$

§ 3A.3. Intercept Form of the Equation of a Plane

Let a plane π does not pass through an origin of coordinates and intercepts the coordinate axes *Ox*, *Oy* and *Oz* at the points $P_1(a,0_1,0)$, $P_2(0,b,0)$ and $P_3(0,0,c)$. Find the equation of such plane.

On substituting the coordinates of the points P_1, P_2 and P_3 into the equation (3A.2.1) we get

$$\begin{vmatrix} x - a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0.$$

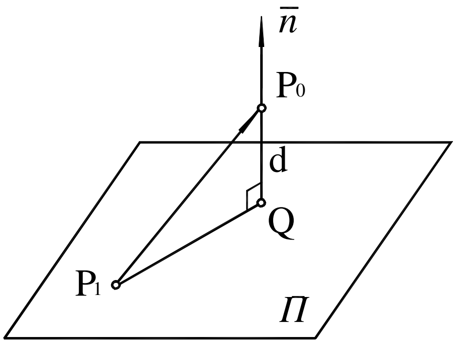
Calculating this determinant we arrive at the following equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (3A.2.2)$$

The equation (3A.2.2) is called the **intercept form of the equation of a plane**.

§ 3A.4. The Distance from a Point to a Plane in Space

It is required to find the distance from the point $P_0(x_0, y_0, z_0)$ to plane π defined by the equation $Ax + By + Cz + D = 0$.



Let $P_1(x_1, y_1, z_1) \in \pi$.Then

$$d = \left|\Pr_{\overline{n}} \overline{P_1P_0}\right| = \frac{\left|\left(\overline{n}, \overline{P_1P_0}\right)\right|}{\left|\overline{n}\right|} =$$

$$= \frac{\left|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)\right|}{\sqrt{A^2 + B^2 + C^2}} =$$

$$= \frac{\left|Ax_0 + By_0 + Cz_0 - (Ax_1 + By_1 + Cz_1)\right|}{\sqrt{A^2 + B^2 + C^2}} = \frac{\left|Ax_0 + By_0 + Cz_0 + D\right|}{\sqrt{A^2 + B^2 + C^2}}.$$

Hence

$$d = \frac{\left|Ax_0 + By_0 + Cz_0 + D\right|}{\sqrt{A^2 + B^2 + C^2}} \qquad (3A.4.1)$$

Example 1. Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\overline{n} = (5, 2, -1)$.

Solution. We use the equation (3A.2.1) to get
$D = -(5 \cdot (-3) + 2 \cdot 0 + (-1) \cdot 7) = 22$.
Hence equation for the plane is $\underline{\underline{5x + 2y - z + 22 = 0}}$.

Example 2. Given the vertices of the triangle: $A(1, 7, 4)$, $B(5, -1, 8)$, $C(5, 2, 3)$. Find an equation of the plane on which the triangle *ABC*.

Solution. On using the equation (3A.2.1) we have

$$\begin{vmatrix} x-1 & y-7 & z-4 \\ 4 & -8 & 4 \\ 4 & -5 & -1 \end{vmatrix} = 0.$$

We expand this determinant to obtain

$$7x + 5y + 3z - 54 = 0.$$

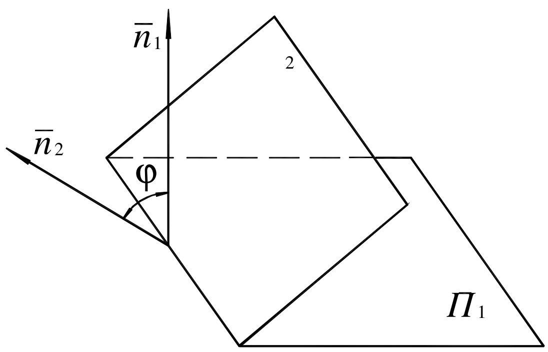
Example 3. Find the distance from the point $P_0(5,1,4)$ to the plane π, given by the equation $4x-3y-3z-14=0$.

Solution. Making use the formula (3A.4.1) we get

$$d=\frac{|4\cdot 5-3\cdot 1-3\cdot 4-14|}{\sqrt{4^2+(-3)^2+(-3)^2}}=\frac{|-9|}{\sqrt{34}}\approx 1.54.$$

§ 3A.5. Angular Relations between Planes

Definition. The angle between two intersecting planes is defined to be the (acute) angle made by their normal vector.



Let

π_1: $A_1x+B_1y+C_1z+D_1=0$,

π_2: $A_2x+B_2y+C_2z+D_2=0$

be the equations of two given planes. It is required to find the angle φ between them and the conditions of the parallelism and the perpendicularity. The normal vectors of these planes are known, that is $\overline{n_1}=(A_1,B_1,C_1)$, $\overline{n_2}=(A_2,B_2,C_2)$, hence

- $\cos\varphi=\pm\dfrac{(\overline{n_1},\overline{n_2})}{|\overline{n_1}|\cdot|\overline{n_2}|}$ (3A.5.1)

- $\pi_1 || \pi_2 \Leftrightarrow \overline{n_1} || \overline{n_2} \Leftrightarrow \dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}$ (3A.5.2)

as the planes are parallel if and only if their normal vectors are collinear. Hence the coordinates of the vector $\overline{n_1}$ are proportional to those of the vector $\overline{n_2}$.

- $\pi_1\perp\pi_2 \Leftrightarrow \overline{n_1}\perp\overline{n_2} \Leftrightarrow A_1A_2+B_1B_2+C_1C_2=0$, (3A.5.3)

Planes π_1 and π_2 are perpendicular to each other if and only if the condition (3A.5.3) is satisfied.

Example. Compute the acute angle between the planes having the equations $9x+8y-12z-85=0$ and $24x-32y+9z+51=0$.

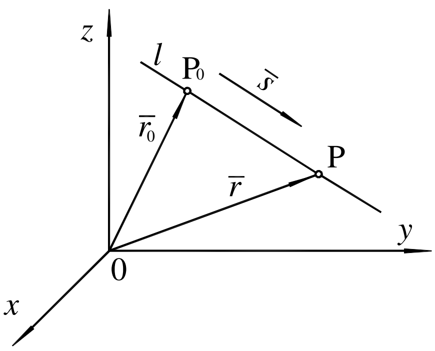
Solution. To use (3A.5.1) calculate $\left|\overline{n_1}\right|=\sqrt{9^2+8^2+12^2}=17$, $\left|\overline{n_2}\right|=\sqrt{24^2+32^2+9^2}=41$. Hence

$$\cos\varphi=\frac{|9\cdot 24+8\cdot(-12)+(-12)\cdot 9|}{17\cdot 41}=\frac{148}{697}\Rightarrow \varphi=\arccos\frac{148}{627}=77.74^{\circ}.$$

The acute angle between the planes is 77.74°.

§ 3A.6. A Straight Line in Space

Definition.. Every non-zero vector lying on the given line or parallel to it is called the **position vector** of that line.



We denote the position vector of a straight line by $\overline{s}$, and coordinates of this vector by *m, n, p*, that is $\overline{s}=(m,n,p)$. Suppose ℓ is a line passing through a point $P_0(x_0,y_0,z_0)$ and lying parallel to a vector $\overline{s}$. Let $P(x,y,z)$ be an arbitrary point of the straight line, hence

$$\overline{P_0P}\,\|\,\overline{s}\Leftrightarrow \overline{P_0P}=t\overline{s}\Leftrightarrow \overline{r}-\overline{r_0}=t\overline{s}\Leftrightarrow \overline{r}=\overline{r_0}+t\overline{s} \qquad (3A.6.1)$$

The equation (3A.6.1) is called the **vector equation** of a straight line. When we write the equation (3A.6.1) in terms of $\overline{i}$, $\overline{j}$, and $\overline{k}$ – components and equate the corresponding components of the two sides, we get three equations involving the parameter *t*:

$$\begin{cases} x=x_0+mt\\ y=y_0+nt\\ z=z_0+pt.\end{cases} \qquad (3A.6.2)$$

We call the equations in (3A.6.2) the **standard parametric equations**.

In the equations (3A.6.2) *t* is regarded as an arbitrarily varying parameter, and *x, y* and *z* vary in such a manner that the point $P(x,y,z)$ moves along the given straight line. The parametric equations of a straight line are conveniently used in cases where it is required to find the point of intersection of the straight line with a plane.

Solving for t the equations (3A.6.2) we get

$$t = \frac{x - x_0}{m},\ t = \frac{y - y_0}{n},\ t = \frac{z - z_0}{p} \tag{3A.6.3}$$

Equating the right-hand sides of (3A.6.3) we obtain

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p} \tag{3A.6.4}$$

The equations (3A.6.4) are called the **canonical equations** of a straight line or the **equations** of a straight line **in the symmetric form.**

In analytic geometry it is often required to write the equations of a straight line two of whose points are given. We shall now find the general solution of this problem letting $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two given arbitrary points of the line.

In order to solve the problem, it is sufficient to note that the vector $\overline{S} = \overline{P_1P_2}$ can be taken as the position vector of the line in question, hence

$$m = x_2 - x_1,\ n = y_2 - y_1,\ p = z_2 - z_1.$$

Assigning to the point P_1 the role played by the point P_0 we have

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \tag{3A.6.5}$$

This is the **two-point form** of the equation of the straight line.

§ 3A.7. Angular Relations between Straight Lines in Space

Let

$$\ell_1 : \frac{x - x_1}{m_1} = \frac{y - y_1}{n_1} = \frac{z - z_1}{p_1}$$

$$\ell_2 : \frac{x - x_2}{m_2} = \frac{y - y_2}{n_2} = \frac{z - z_2}{p_2}$$

be the equations of two given straight lines. The angle φ between these lines is revered to the angle between their position vectors. Hence we have

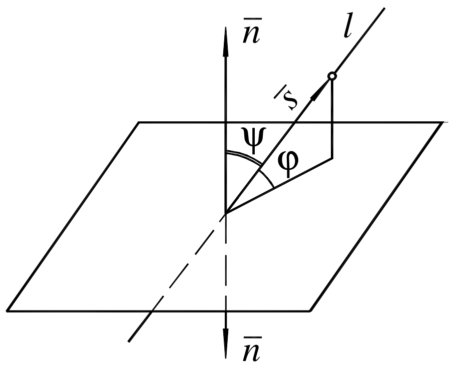
• $\cos\varphi = \pm\dfrac{\left(\overline{S}_1, \overline{S_2}\right)}{\left|\overline{S_1}\right| \cdot \left|\overline{S_2}\right|}$, (3A.7.1)

• $\ell_1 \,|\,|\, \ell_2 \Leftrightarrow \overline{S_1} \,|\,|\, \overline{S_2} \Leftrightarrow \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} = \dfrac{p_1}{p_2}$, (3A.7.2)

• $\ell_1 \perp \ell_2 \Leftrightarrow \overline{S_1} \perp \overline{S_2} \Leftrightarrow \left(\overline{S_1}, \overline{S_2}\right) = 0 \Leftrightarrow m_1 m_2 + n_1 n_2 + p_1 p_2 = 0$. (3A.7.3)

§ 3A.8. Parallelism and Perpendicularity of a Line and a Plane

Let



$$\ell : \frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}$$

$\pi : Ax + By + Cz + D = 0$ be given.

An **angle φ between a straight line ℓ and a plane π** is an angle between this line and its projection on the plane.

$\psi = 90^0 \pm \varphi \Rightarrow \cos\psi = \mp \sin\varphi$, or

$$\sin\varphi = \left|\cos\psi\right| = \cos\left(\overline{s}, ^\wedge \overline{n}\right)$$

• $\sin\varphi = \left|\dfrac{\left(\overline{n}, \overline{s}\right)}{\left|\overline{n}\right| \cdot \left|\overline{\mathrm{s}}\right|}\right|$, (3A.8.1)

• $\pi \,|\,|\, \ell \Leftrightarrow \overline{n} \perp \overline{s} \Leftrightarrow Am + Bn + Cp = 0$, (3A.8.2)

• $\pi \perp \ell \Leftrightarrow \overline{n} \,|\,|\, \overline{s} \Leftrightarrow \dfrac{A}{m} = \dfrac{B}{n} = \dfrac{C}{p}$. (3A.8.3)

§ 3A.9. A Straight Line in Plane

A straight line in a plane may be determined by a normal vector or a position vector. So for a straight line we get two groups of formulas. The first group is associated with a normal vector. The second group is associated with a position vector.

• 1. The first group of equations

1). $A(x-x_0)+B(y-y_0)+C=0$ (3A.9.1)

This is the equation of a straight line through the point $P_0(x_0, y_0)$ with the normal vector $\bar{n}=(A, B)$.

2). **A general equation**

$$Ax+By+C=0 \qquad (3A.9.2)$$

3). A **distance from a point** $P_0(x_0, y_0)$ **to a straight line**

$$d=\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}} \qquad (3A.9.3)$$

• 2. The second group of equations

1). **A vector equation**

$$\bar{r}=\overline{r_0}+t\bar{s}, \qquad (3A.9.4)$$

where

$\bar{r}=(x, y)$, $\overline{r_0}=(x_0, y_0)$, $\bar{s}=(m, n)$.

2). **A parametric equations**

$$\begin{cases} x=x_0+mt \\ y=y_0+nt \end{cases} \qquad (3A.9.5)$$

3). **A canonical equation**

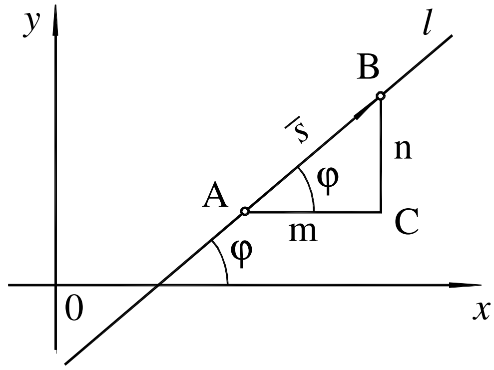
$$\frac{x-x_0}{m}=\frac{y-y_0}{n} \qquad (3A.9.6)$$

4). **A two-points equation**

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad (3A.9.7)$$

§ 3A.10. The Third Group of Equations of a Straight Line in a Plane

The third group is associated with a slope k.



The line ℓ forms the angle φ with the positive direction of x-axis. This angle is called an angle of inclination with respect to x-axis. The tangent of an angle of inclination of a straight line is called the slope of this line and denoted by k: $k = \tan\varphi$.

It follows from the triangle ΔABC that $k = \dfrac{n}{m}$.

Using a canonical equation (3A.9.6) we get

$$y - y_0 = k(x - x_0) \qquad (3A.10.1)$$

This equation is called the **point – slope equation** of a straight line.

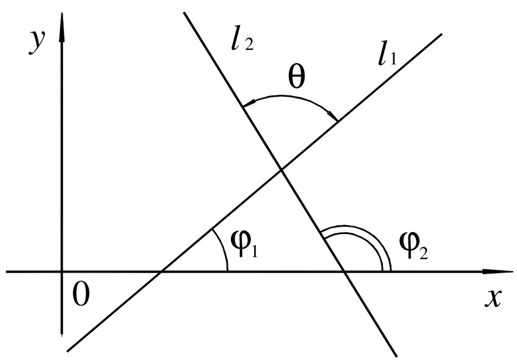
Rearranging the terms of the equation (3A.10.1) we have

$$y = kx + b, \qquad (3A.10.2)$$

where $b = y_0 - kx_0$, b is called the y-intercept of the line, and the equation (3A.10.2) is called the **slope-intercept equation** of a straight line.

Now we find an angle between two straight lines.

Definition. The least angle through which it is necessary to rotate the straight line ℓ_1 anticlockwise to reach the coincidence with the second line ℓ_2 is called the angle between these straight lines.



$$\tan\theta = \tan(\varphi_2 - \varphi_1) = \frac{\tan\varphi_2 - \tan\varphi_1}{1 + \tan\varphi_1 \tan\varphi_2};$$

$$\tan\theta = \frac{k_2 - k_1}{1 + k_1 k_2} \tag{3A.10.3}$$

§ 3A.11. Conditions of Parallelism and Perpendicularity of Sloping Straight Lines

Two straight lines are parallel if and only if their slopes are equal. In fact:

$\ell_1 \| \ell_2 \Leftrightarrow \theta = 0 \Leftrightarrow \tan\theta = 0 \Leftrightarrow k_1 = k_2$, thus

$$\bullet\ \ell_1 \| \ell_2 \Leftrightarrow k_1 = k_2 \tag{3A.11.1}$$

Two straight lines are perpendicular if and only if their slopes relate to each other as $k_2 = -\frac{1}{k_1}$. In fact:

$$\ell_1 \perp \ell_2 \Leftrightarrow \theta = 90^0 \Leftrightarrow \cot\theta = 0 \Leftrightarrow \frac{1 + k_2 k_1}{k_2 - k_1} = 0 \Leftrightarrow 1 + k_1 k_2 = 0 \Leftrightarrow k_2 = -\frac{1}{k_1}$$

$$\bullet\ \ell_1 \perp \ell_2 \Leftrightarrow k_2 = -\frac{1}{k_1} \tag{3A.11.2}$$

§ 3A.12. Solution of Problems

Example 1. The equations $\frac{x-1}{1} = \frac{y-2}{0} = \frac{z-3}{2}$ define the straight line in space passing through the point $(1,2,3)$ in the direction of the vector $(1,0,2)$. These equations can be replaced by the following equivalent ones:

$$\begin{cases} y - 2 = 0 \cdot (x-1), \\ 2(x-1) = z - 3, \end{cases} \text{ i.e. } \begin{cases} y = 2, \\ z = 2x + 1. \end{cases}$$

Thus the straight line under consideration is an intersection of two planes defined by the equations $y = 2$ and $z = 2x + 1$.

Example 2. Let a straight line ℓ be defined by equations

$$\begin{cases} x - 3y + 2z - 4 = 0, \\ 2x + y - 5z - 15 = 0. \end{cases}$$

Find a canonical equation of this line.

Solution.

The line ℓ is an intersection of two planes $\pi_1 : x - 3y + 2z = 0$ and $\pi_2 : 2x + y - 5z - 15 = 0$, hence its every point belongs to each of these planes. Find one of them. For example, $P_0(x_0, y_0, z_0)$ if $z_0 = 0$. Then x_0 and y_0 satisfy the system of equations

$$\begin{cases} x_0 - 3y_0 = 4, \\ 2x_0 + y_0 = 15. \end{cases}$$

Solving this system we get $x_0 = 7$, $y_0 = 1$, and $P_0(7,1,0)$.

Next we find some vector parallel to the straight line ℓ. Normal vectors of the planes are $\overline{n_1} = (1,-3,2)$, $\overline{n_2} = (2,1,-5)$. Then the vector $\overline{s} = [\overline{n_1}, \overline{n_2}]$ is parallel to the straight line ℓ and therefore is its position vector:

$$\overline{s} = [\overline{n_1}, \overline{n_2}] = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ 1 & -3 & 2 \\ 2 & 1 & -5 \end{vmatrix} = 13\overline{i} + 9\overline{j} + 7\overline{k}.$$

Using the equations (3A.6.4) we have the canonical equations of the straight line ℓ:

$$\ell : \frac{x-7}{13} = \frac{y-1}{9} = \frac{z}{7}.$$

Example 3. The straight lines ℓ_1 and ℓ_2 are given by the equations:

$$\ell_1 : \frac{x-3}{1} = \frac{y+5}{4} = \frac{z}{5},$$
$$\ell_2 : \frac{x+8}{-5} = \frac{y-9}{0} = \frac{z+10}{1}.$$

Find out whether these straight lines are parallel or perpendicular.

Solution. The position vectors of the straight lines ℓ_1 and ℓ_2 are

$$\overline{s_1} = (1,4,5), \ \overline{s_2} = (-5,0,1).$$

Using the condition (3A.7.2) we get $\frac{1}{-5} \neq \frac{4}{0} \neq \frac{5}{1}$. Hence the vectors $\overline{s_1}$ and $\overline{s_2}$ are not collinear and the straight lines ℓ_1 and ℓ_2 are not parallel.

Let us verify that the condition of perpendicularity of these straight lines (3A.7.3) is satisfied.

We have $\left(\overline{S_1}, \overline{S_2}\right) = 1 \cdot (-5) + 4 \cdot 0 + 5 \cdot 1 = 0$. This means that the straight lines ℓ_1 and ℓ_2 are perpendicular.

Example 4. A straight line passing through the point $P(-2,3)$ forms with the *x*-axis the angle 135^0. Find an equation of this straight line.

Solution. An equation of this straight line we seek in the form $y = kx + b$.

1). The slope of this straight line is $k = \tan 135^0 = -1$.

2). The straight line $y = -x + b$ passes through the point $P(-2,3)$, therefore its coordinates $x = -2, y = 3$ satisfy the equation of this line, that is $3 = -(-2) + b \Rightarrow b = 1$. Hence the equation of the straight line has the form $\underline{y = -x + 1.}$

B. SECOND ORDER CURVES

§ 3B.1. Parabolas

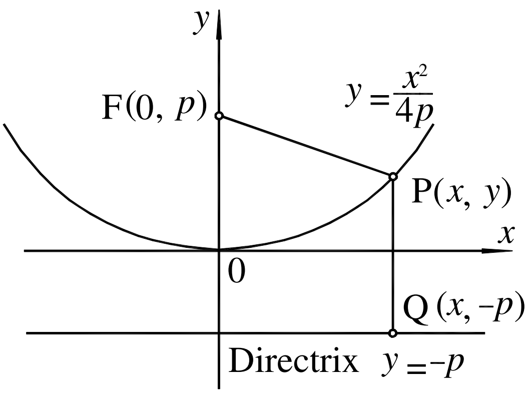


Fig. 3B.1

Definition. A **parabola** is the set of points in a plane that are equidistant from a given fixed point and fixed line in this plane. The fixed point is the parabola's **focus** $F(0, p)$. The fixed line is the parabola's **directrix** $y = -p$.

In the notation of the figure, a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$. From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \\ = \sqrt{x^2 + (y-p)^2},$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}\ .$$

When we equate these expressions, square and simplify, we get standard equation of this parabola:

$$y = \frac{x^2}{4p} \qquad (3B.1.1)$$

This equation reveals the parabola's symmetry about the y-axis. We call the y-axis the **axis of the parabola** (short for "axis of symmetry").

The point where a parabola crosses its axis, midway between the focus and directrix, is called the **vertex of the parabola**. The vertex of the parabola $y = \frac{x^2}{4p}$ lies at the origin. The number p is the focal length of the parabola, and $4p$ **is the width of the parabola at the focus**.

Example 3B.1.1 Find the focus and directix of the parabola $y = \frac{x^2}{8}$.

Solution.

Step 1. Find the value of p in the standard equation: $y = \frac{x^2}{8}$ is $y = \frac{x^2}{4p}$ with $p = 2$.

Step 2. Find the focus and directrix for the value of $p = 2$:

Focus: $F(0, p) \Rightarrow F(0,2)$.

Directrix: $y = -p \Rightarrow y = -2$.

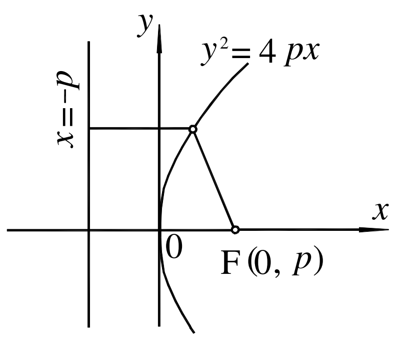


Fig.3B.2

If we interchange x and y in the formula $y = \frac{x^2}{4p}$, we obtain the equation

$$y^2 = 4px \quad (p > 0) \qquad (3B.1.1)$$

With the role of x and y now reversed, the graph is a parabola whose axis is the x-axis. The vertex still lies at the origin. The parabola opens to the right.

The chief application of parabolas involves their use as reflectors of light and radio waves.

Rays originating at a parabola's focus are reflected out of the parabola parallel to the parabola's axis (in *Fig.3.B.2*, the *x*-axis). Similarly, rays coming in parallel to the axis are reflected toward the focus. This property is used in parabolic mirrors and telescopes, in automobile headlamps, in spotlights of all kinds, radar and microwave antennas, and in solar collectors. Parabolas are also used in bridge constructions, wind tunnel photography, and submarine tracking.

§ 3B.2. Ellipses

Definition. An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse

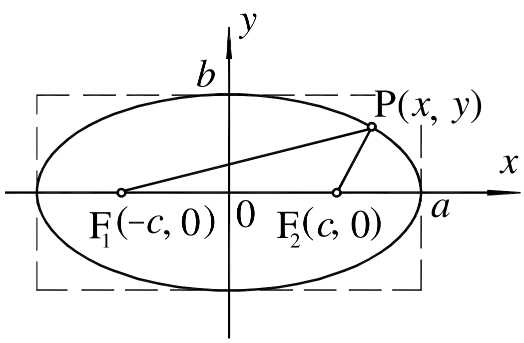


Fig.3.B.3

The line through the foci of an ellipse is the **ellipse's focal axis**. The point of the axis halfway between the foci is the ellipse's **center**. The points where the focal axis crosses the ellipse are the ellipse's **vertices**.

If the foci are $F_1(-c,0)$ and $F_2(c,0)$, and the sum of the distances $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \qquad (3B.2.1)$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad (3B.2.2)$$

Since the sum $PF_1 + PF_2$ is greater than the length F_1F_2 (triangle inequality for triangle PF_1F_2), the number $2a$ is greater than $2c$. Accordingly, a is greater than c and the number $a^2 - c^2$ in equation (3B.2.2) is positive.

If $b = \sqrt{a^2 - c^2}$, then equation (3B.2.2) takes the more compact form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ (standard equation)} \qquad (3B.2.3)$$

The **major axis** of the ellipse described by equation (3B.2.3) is the line segment of length $2a$ joining the points $(a,0)$ and $(-a,0)$.

The **minor axis** of the ellipse described by equation (3B.2.3) is the line segment of length $2b$ joining the points $(0,b)$ and $(0,-b)$. The number a itself is called the **semimajor axis** and the number b the **semiminor axis**. The number c, which can be found as $c = \sqrt{a^2 - b^2}$ is the **center-to-focus distance** of the ellipse.

If we keep a fixed and vary c over the interval $0 \le c \le a$ the resulting ellipses will vary in shape. They are circles if $c = 0$ (so that $a = b$) and flatten as c increases. In the extreme case $c = a$, the foci and vertices overlap and the ellipse degenerates into a line segment. We use the ratio of c to a to describe the various shapes the ellipse can take. We call this ratio the ellipse's eccentricity.

Definition. The **eccentricity** of the ellipse is the number

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} \qquad (3B.2.4)$$

The planets in the solar system revolve around the sun in elliptical orbits with the sun at one focus. Most of the planets, including Earth $(e = 0.02)$, have orbits that are circular. Pluto, however, has a fairly eccentric orbit, with $e = 0.25$, as does Mercury, with $e = 0.21$. Other members of the solar system have orbits that are even more eccentric. Icarus, an asteroid about one mile wide that revolves around the sun every 409 Earth days, has an orbital eccentricity of 0.83.

Ellipses appear in airplane wings (British Spitfire) and sometimes in gears designed by racing bicycles.

Stereo systems often have elliptical styli, and water pipes are sometimes designed with elliptical cross sections to allow for expansion when the water freezes.

The triggering mechanisms in some lasers are elliptical, and stones on a beach become more and more elliptical as they are ground down by waves. There are also applications of ellipses to fossil formation. The ellipsolith, once thought to be a separate species, is now known be an elliptically deformed nautilus.

Example 3B.2.1. Find the standard-form equation of the ellipse with foci $(0, \pm 3)$ and vertices $(0, \pm 4)$.

Solution.

The standard-form equation for an ellipse with foci $(0, \pm c)$ and vertices $(0, \pm a)$ is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $c = \sqrt{a^2 - b^2}$. In the ellipse at hand $c = 3$ and $a = 4$, so $3 = \sqrt{4^2 - b^2} \Rightarrow 9 = 16 - b^2 \Rightarrow b^2 = 7$. The equation we seek is

$$\frac{x^2}{(\sqrt{7})^2} + \frac{y^2}{4^2} = 1.$$

Example 3B.2.2. The orbit of Halley's comet is an ellipse 36.18 astronomical units long by 9.12 astronomical units wide. Find its eccentricity.

Solution.

One astronomical unit is the semimajor axis of the earth's orbit, about 92,600,000 miles. It's eccentricity is

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{(36.18/2)^2 - (9.12/2)^2}}{36.18/2} =$$

$$= \frac{\sqrt{(18.09)^2 - (4.56)^2}}{18.09} = 0.97. \text{ (Rounded, with a calculator).}$$

§ 3B.3. Hyperbolas

Definition. A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

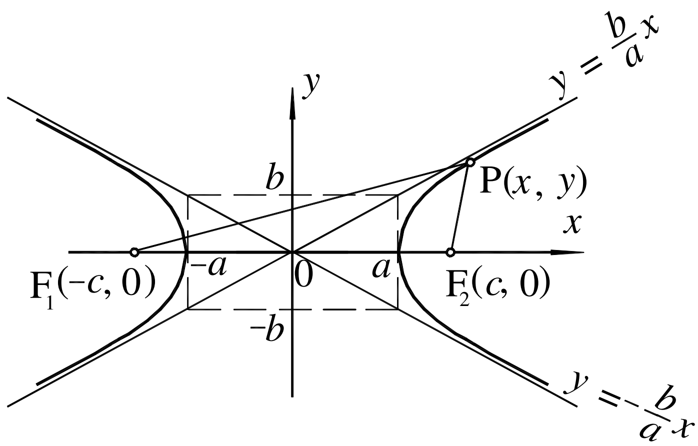


Fig.3.B.4

Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$. If the foci are $F_1(-c,0)$ and $F_2(c,0)$ and the constant difference is 2*a*, then a point $P(x, y)$lies on the hyperbola if and only if

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \qquad \text{(3B.3.1)}$$

To simplify the equation (3B.3.1), we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \qquad \text{(3B.3.2)}$$

So far, this looks just like the equation for an ellipse. But now $a^2 - c^2$ is negative because 2*a*, being the difference of two sides of triangle PF_1F_2, is less than 2*c*, the third side.

The algebraic steps taken to arrive at equation (3B.3.2) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $o < a < c$ also satisfies the equation (3B.3.1). Thus, a point lies on the hyperbola if and only if its coordinates satisfy the equation (3B.3.2).

If we let *b* denote the positive square root of $c^2 - a^2$,

$$b = \sqrt{c^2 - a^2}\ , \qquad \text{(3B.3.3)}$$

then $a^2 - c^2 = -b^2$ and the equation (3B.3.2) takes the more compact form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (Standard equation)} \qquad \text{(3B.3.4)}$$

The line through the foci of a hyperbola is the hyperbola's **focal axis**.

The point on the axis halfway between the foci is the hyperbola's **center**.

The points where the focal axis crosses the hyperbola are the hyperbola's **vertices**.

If the distance between a curve and some fixed line may approach zero as the a point of curve moves farther and farther from the origin then this line is called an **asymptote** of the curve.

The hyperbola (3B.3.4) has two asymptotes, the lines

$$y = \pm \frac{b}{a} x \qquad (3B.3.5)$$

Definition. The **eccentricity** of the hyperbola (3B.3.4) is the number

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} \qquad (3B.3.6)$$

In both ellipse and hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $\frac{c}{a} = \frac{2c}{2a}$). In an ellipse, the foci are closer together than the vertices and the ratio is less than 1. In a hyperbola, the foci are farther apart than the vertices and the ratio is greater than 1.

Hyperbolic paths arise in Einstein's theory of relativity and form the basis for the (unrelated) LORAN radio navigation system. (LORAN is short for "long range navigation".) Hyperbolas also form the basis for a new system the Burlington Northern Railroad is developing for using synchronized electronic signals from satellites to track freight trains.

Example.

Find an equation for the hyperbola with asymptotes $y = \pm \frac{4}{3} x$ and foci $(\pm 10, 0)$.

Solution.

The standard form equation for a hyperbola with foci $(\pm c, 0)$ **on the *x*-axis is** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$**, where** $c = \sqrt{a^2 + b^2}$**. From the asymptote equation** $y = \pm \frac{b}{a} x$**, we learn that** $\frac{b}{a} = \frac{4}{3}$ **, or** $b = \frac{4}{3} a$**.**

Hence, $c^2 = a^2 + b^2 = a^2 + \frac{16}{9} a^2 = \frac{25}{9} a^2 \Rightarrow a^2 = \frac{9}{25} c^2 = \frac{9}{25} \cdot 10^2 = 36$**,** $b^2 = c^2 - a^2 = 100 - 36 = 64$. (The foci are $(\pm 10, 0)$, so $c = 10$ and $c^2 = 100$.

The equation we seek is

$$\frac{x^2}{36} - \frac{y^2}{64} = 1.$$

Miscellaneous Problems

Exercise 1.

Let points $M(a;b;c)$, $N(a+b;c;b)$, $P(-c;a;-b)$, $K(-a;-b;c)$,

straight lines $\ell_1 : \dfrac{x-a}{\alpha} = \dfrac{y+b}{b} = \dfrac{z-b}{c}$; $\quad \ell_2 : \dfrac{x+a}{a} = \dfrac{y-b}{\beta} = \dfrac{z+b}{\gamma}$,

$$\ell_3 : \begin{cases} x = -t + a \\ y = at + b \\ z = -bt + c \end{cases},$$

and planes

$\pi_1 : \alpha x + abcy + (c - a + 1)z + b = 0$, $\pi_2 : cx + \beta y + \gamma z - c = 0$,

$\pi_3 : (a+b)x + (b - c + 3)y + bcz - a = 0$,

be given.

I. **Find equation of**

1. the plane *MNP*;
2. the plane π through the point *K* perpendicular to *MN*;
3. the plane π through the point *M* parallel to plane *XOZ*;
4. the plane π through the point *N* perpendicular to *Y*-axis;
5. the plane π through the point *P* parallel to plane π_3;
6. the plane π through the point *K* perpendicular to ℓ_3;
7. the plane π through the point *M* and ℓ_3;
8. the straight line ℓ through the point *M* parallel to *PN*;
9. the straight line ℓ through the point *N* perpendicular to π_3, and find the angle between this line ℓ and ℓ_3;
10. the straight line ℓ through the point *P* parallel to ℓ_3,
11. the straight line ℓ through the point *M* parallel to *z*-axis;
12. the straight line ℓ through the points *M* and *K*.

II. **Find**

1. the intersection point of the straight line ℓ_3 and the plane π_3;
2. the angle between the plane *MNP* and π_3;
3. the distance from the point *K* to the plane π_3;
4. the angle between the straight lines *MK* and ℓ_3;
5. direction vector of the intersection line of the planes *MNP* and π_3.

III. **Find** α, β, γ such that

1. $\ell_2 || \ell_3$; 2. $\ell_1 \perp \ell_3$ 3. $\ell_3 \perp \pi_2$;
4. $\ell_3 \subset \pi_2$; 5. $\pi_1 \perp \pi_3$ 6. $\pi_2 || \pi_3$.

Exercise 2.

Given the triangle *MNP* where $M(a;b)$, $N(a+b;c)$, $P(-c;a)$. Find

1. a slope of *MN*;
2. an equation of median *MK*;
3. an equation of altitude *MH*;
4. the angle *KMH* through the slopes of *MK* and *MH*.

Exercise 3.

Classify each of the following second-degree equations as representing a circle, an ellipse, a parabola, or a hyperbola. Draw them.

1. $ax^2 + ay^2 + x - y = 3$
2. $2bx^2 - by^2 - 2x + 3y = 6$
3. $x^2 + 4y^2 - 3ax + by = 6$
4. $y^2 + bx - cy = 3$
5. $x^2 + ax - cy = 3$
6. $y^2 - x^2 - 2ax + 3by = 6$

a – the first letter of your surname
b – the first letter of your name
c – the first letter of your patronymic

1	2	3	4	5	6	7	8	9
A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	